Continuous Location Problems and DC optimization

Emilio Carrizosa
Universidad de Sevilla, Spain

SYM-OP-IS 2009

Continuous Location Problems and DC optimization

Emilio Carrizosa
Universidad de Sevilla, Spain

SYM-OP-IS 2009

- Ongoing research at our researh group, http://www-en.us.es/gioptim

- R. Blanquero, E. Carrizosa, P. Hansen: Locating Objects in the Plane Using Global Optimization Techniques. Mathematics of OR. To appear.

- R. Blanquero, E. Carrizosa: Continuous Location Problems And Big Triangle Small Triangle: Constructing Better Bounds. JOGO. To appear.

- R. Blanquero, E. Carrizosa: A DC Biobjective Location Model. JOGO 23 (2002).
- E. Carrizosa: An Optimal Bound for D.C. Programs With Convex Constraints. MMOR 54 (2001).
- R. Blanquero, E. Carrizosa, E. Conde: Finding Gm-Estimators With Global Optimization Techniques. JOGO 21 (2001).
- R. Blanquero, E. Carrizosa: On Covering Methods for D.C. Optimization. JOGO 18 (2000).
- R. Blanquero, E. Carrizosa: Optimization of the Norm of a Vector-Valued DC Function and Applications. JOTA 107 (2000).

DC functions

Tuy, 1998

Actually most functions encountered in practice are dc

$f : \Omega \longrightarrow \mathbb{R}$ is said to be dc (difference of convex functions) on the convex set Ω if $f + h$ is convex on $\Omega \subset \mathbb{R}^n$ for some $h : \Omega \longrightarrow \mathbb{R}$ convex on Ω.

$f : \Omega \longrightarrow \mathbb{R}$ is said to be dc (difference of convex functions) on the convex set Ω if $g, h : \Omega \longrightarrow \mathbb{R}$ exist such that both g, h are convex on Ω and $f = g - h$.

DC functions

Tuy, 1998

Actually most functions encountered in practice are dc

$f : \Omega \longrightarrow \mathbb{R}$ is said to be dc (difference of convex functions) on the convex set Ω if $f + h$ is convex on $\Omega \subset \mathbb{R}^n$ for some $h : \Omega \longrightarrow \mathbb{R}$ convex on Ω.

$f : \Omega \longrightarrow \mathbb{R}$ is said to be dc (difference of convex functions) on the convex set Ω if $g, h : \Omega \longrightarrow \mathbb{R}$ exist such that both g, h are convex on Ω and $f = g - h$.

DC functions

Tuy, 1998

Actually most functions encountered in practice are dc

$f : \Omega \longrightarrow \mathbb{R}$ is said to be dc (difference of convex functions) on the convex set Ω if $f + h$ is convex on $\Omega \subset \mathbb{R}^n$ for some $h : \Omega \longrightarrow \mathbb{R}$ convex on Ω.

$f : \Omega \longrightarrow \mathbb{R}$ is said to be dc (difference of convex functions) on the convex set Ω if $g, h : \Omega \longrightarrow \mathbb{R}$ exist such that both g, h are convex on Ω and $f = g - h$.

DC decompositions

dc decomposition

$f = g - h$

If $f = (f + h) - h$ is a dc decomposition of the dc function f, then, any convex function k yields another dc decomposition, namely $f = (f + h + k) - (h + k)$.

DC decompositions

dc decomposition

$f = g - h$

If $f = (f + h) - h$ is a dc decomposition of the dc function f, then, any convex function k yields another dc decomposition, namely $f = (f + h + k) - (h + k)$.

A few references

- P. Hartman, *On functions representable as a difference of convex functions*, Pacific J. Math. **9** (1959), 707–713.
- J. B. Hiriart-Urruty, *Generalized Differentiability, Duality, and Optimization for Problems Dealing with Differences of Convex Functions*, Lecture Notes in Economics and Mathematical Systems 256 (J. Ponstein, ed.), Springer Verlag, Berlin, 1986.
- R. Horst and N. V. Thoai, *DC Programming: Overview*, J. Optim. Theory Appl. **103** (1999), 1–43.

A few references (II)

- R. Horst and H. Tuy, *Global Optimization. Deterministic Approaches*, Springer-Verlag, Berlin, 1996.
- H. Tuy, *D.C. Optimization: Theory, Methods and Algorithms*, Handbook of Global Optimization (R. Horst and P. M. Pardalos, eds.), Kluwer Academic Publishers, Dordrecht, 1995.
- H. Tuy, *Convex Analysis and Global Optimization*, Kluwer Academic Publishers, Dordrecht, 1998.

Properties of dc functions

If $f \in \mathcal{C}^2(\mathbb{R}^n)$, then f is dc on any compact convex Ω.

Why?

Write $f(x)$ as $(f(x) + \tau\|x\|^2) - \tau\|x\|^2$ for τ big enough.

Suppose Ω is closed or open. A locally dc function on Ω is also globally dc on Ω.

This result does not tell us how to obtain a dc decomposition!

Properties of dc functions

If $f \in \mathcal{C}^2(\mathbb{R}^n)$, then f is dc on any compact convex Ω.

Why?

Write $f(x)$ as $\left(f(x) + \tau\|x\|^2\right) - \tau\|x\|^2$ for τ big enough.

Suppose Ω is closed or open. A locally dc function on Ω is also globally dc on Ω.

This result does not tell us how to obtain a dc decomposition!

Properties of dc functions

If $f \in \mathcal{C}^2(\mathbb{R}^n)$, then f is dc on any compact convex Ω.

Why?

Write $f(x)$ as $\left(f(x) + \tau\|x\|^2\right) - \tau\|x\|^2$ for τ big enough.

Suppose Ω is closed or open. A locally dc function on Ω is also globally dc on Ω.

This result does not tell us how to obtain a dc decomposition!

Properties of dc functions

If $f \in \mathcal{C}^2(\mathbb{R}^n)$, then f is dc on any compact convex Ω.

Why?

Write $f(x)$ as $(f(x) + \tau\|x\|^2) - \tau\|x\|^2$ for τ big enough.

Suppose Ω is closed or open. A locally dc function on Ω is also globally dc on Ω.

This result does not tell us how to obtain a dc decomposition!

Composition

Let $\Omega_1 \subset \mathbb{R}^n, \Omega_2 \subset \mathbb{R}^m$ convex, with Ω_1 : open or closed and Ω_2 : open. If $F_1 : \Omega_1 \longrightarrow \Omega_2$ and $F_2 : \Omega_2 \longrightarrow \mathbb{R}^k$ are componentwise dc, then $F_2 \circ F_1 : \Omega_1 \longrightarrow \mathbb{R}^k$ is componentwise dc.

For Ω_1, Ω_2 as above, if F_1 is dc and F_2 is $\mathcal{C}^2$, then $F_2 \circ F_1$ is dc. In particular,

- the product of dc functions on a convex closed or open set Ω is dc
- if f is positive and dc on a convex closed or open set Ω, then $1/f$ is also dc.

Composition

Let $\Omega_1 \subset \mathbb{R}^n, \Omega_2 \subset \mathbb{R}^m$ convex, with Ω_1 : open or closed and Ω_2 : open. If $F_1 : \Omega_1 \longrightarrow \Omega_2$ and $F_2 : \Omega_2 \longrightarrow \mathbb{R}^k$ are componentwise dc, then $F_2 \circ F_1 : \Omega_1 \longrightarrow \mathbb{R}^k$ is componentwise dc.

For Ω_1, Ω_2 as above, if F_1 is dc and F_2 is $\mathcal{C}^2$, then $F_2 \circ F_1$ is dc. In particular,

- the product of dc functions on a convex closed or open set Ω is dc
- if f is positive and dc on a convex closed or open set Ω, then $1/f$ is also dc.

Obtaining dc decompositions

Tuy, 1998

The problem of effective dc representation is far from trivial

Horst-Tuy, 1990

. . . however, in many special cases of interest, the exact dc decomposition is already given or easily found

Obtaining dc decompositions

Tuy, 1998

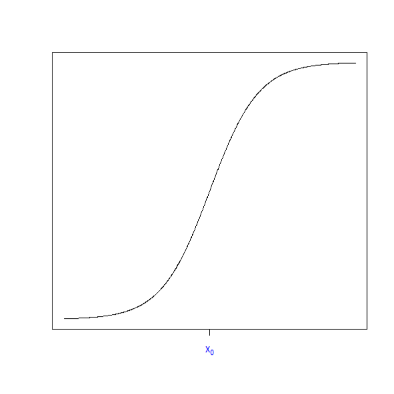
The problem of effective dc representation is far from trivial

Horst-Tuy, 1990

. . . however, in many special cases of interest, the exact dc decomposition is already given or easily found

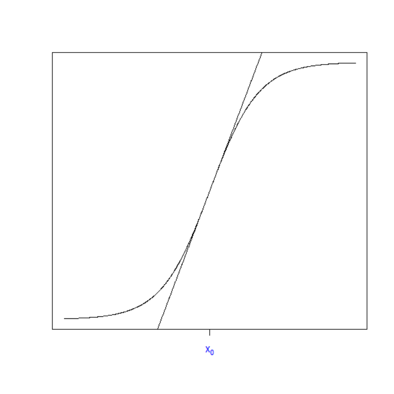
Obtaining dc decompositions

$f : \mathbb{R} \longrightarrow \mathbb{R}$, smooth, convex in $(-\infty, x_0)$, concave in $(x_0, +\infty)$



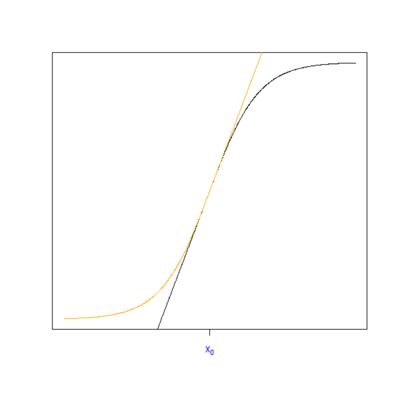
Obtaining dc decompositions

$f : \mathbb{R} \longrightarrow \mathbb{R}$, smooth, convex in $(-\infty, x_0)$, concave in $(x_0, +\infty)$



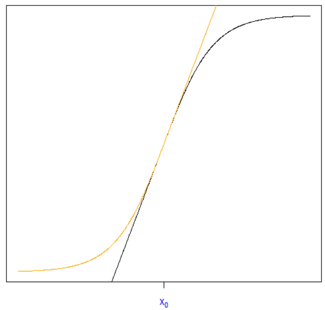
Obtaining dc decompositions

$f : \mathbb{R} \longrightarrow \mathbb{R}$, smooth, convex in $(-\infty, x_0)$, concave in $(x_0, +\infty)$



Obtaining dc decompositions

$f : \mathbb{R} \longrightarrow \mathbb{R}$, smooth, convex in $(-\infty, x_0)$, concave in $(x_0, +\infty)$



$f = g - h$

$g(x) = \ldots$

$$\begin{cases} f(x), & \text{if } x \leq x_0 \\ f(x_0) + f'(x_0)(x - x_0), & \text{if } x > x_0 \end{cases}$$

$h(x) = g(x) - f(x) = \ldots$

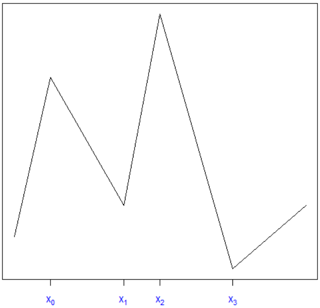
$$\begin{cases} 0, & \text{if } x \leq x_0 \\ f(x_0) + f'(x_0)(x - x_0) - f(x), & \text{if } x > x_0 \end{cases}$$

Obtaining dc decompositions

$f : \mathbb{R} \longrightarrow \mathbb{R}$, piecewise linear and continuous

$f(x) = \ldots$

$$\begin{cases} m_0 x + n_0, & \text{if } x \le x_0 \\ m_1 x + n_1, & \text{if } x_0 \le x \le x_1 \\ \vdots & \vdots \\ m_r x + n_r, & \text{if } x_{r-1} \le x_r \\ m_{r+1} x + n_{r+1}, & \text{if } x_r \le x \end{cases}$$



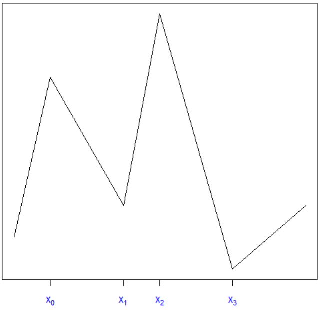
Obtaining dc decompositions

$f : \mathbb{R} \longrightarrow \mathbb{R}$, piecewise linear and continuous

$g(x) = \dots$

$$\begin{cases} a_0 x + b_0, & \text{if } x \leq x_0 \\ a_1 x + b_1, & \text{if } x_0 \leq x \leq x_1 \\ \vdots & \vdots \\ a_r x + b_r, & \text{if } x_{r-1} \leq x_r \\ a_{r+1} x + b_{r+1}, & \text{if } x_r \leq x \end{cases}$$

$h(x) = g(x) - f(x)$



Obtaining dc decompositions

$f : \mathbb{R} \longrightarrow \mathbb{R}$, piecewise linear and continuous

$g(x) = \ldots$

$$\begin{cases} a_0 x + b_0, & \text{if } x \leq x_0 \\ a_1 x + b_1, & \text{if } x_0 \leq x \leq x_1 \\ \vdots & \vdots \\ a_r x + b_r, & \text{if } x_{r-1} \leq x_r \\ a_{r+1} x + b_{r+1}, & \text{if } x_r \leq x \end{cases}$$

$h(x) = g(x) - f(x)$

Convexity of g, h

$$\begin{aligned} a_i &\leq a_{i+1} \\ a_i - m_i &\leq a_{i+1} - m_{i+1} \\ a_i x_i + b_i &= a_{i+1} x_i + b_{i+1} \end{aligned}$$

$$\begin{aligned} a_0 &= 0 \\ a_i &= a_{i-1} + \max\{0, m_i - m_{i-1}\} \\ b_0 &= 0 \\ b_i &= b_{i-1} + (a_{i-1} - a_i) x_{i-1} \end{aligned}$$

Obtaining dc decompositions

$f : \mathbb{R} \longrightarrow \mathbb{R}$, piecewise linear and continuous

$g(x) = \dots$

$$\begin{cases} a_0 x + b_0, & \text{if } x \leq x_0 \\ a_1 x + b_1, & \text{if } x_0 \leq x \leq x_1 \\ \vdots & \vdots \\ a_r x + b_r, & \text{if } x_{r-1} \leq x_r \\ a_{r+1} x + b_{r+1}, & \text{if } x_r \leq x \end{cases}$$

$h(x) = g(x) - f(x)$

Convexity of g, h

$$\begin{array}{rcl} a_i & \leq & a_{i+1} \\ a_i - m_i & \leq & a_{i+1} - m_{i+1} \\ a_i x_i + b_i & = & a_{i+1} x_i + b_{i+1} \end{array}$$

$$\begin{array}{rcl} a_0 & = & 0 \\ a_i & = & a_{i-1} + \max\{0, m_i - m_{i-1}\} \\ b_0 & = & 0 \\ b_i & = & b_{i-1} + (a_{i-1} - a_i) x_{i-1} \end{array}$$

Obtaining dc decompositions

$f : \mathbb{R} \longrightarrow \mathbb{R}$, piecewise linear and continuous

$g(x) = \ldots$

$$\begin{cases} a_0 x + b_0, & \text{if } x \leq x_0 \\ a_1 x + b_1, & \text{if } x_0 \leq x \leq x_1 \\ \vdots & \vdots \\ a_r x + b_r, & \text{if } x_{r-1} \leq x_r \\ a_{r+1} x + b_{r+1}, & \text{if } x_r \leq x \end{cases}$$

$h(x) = g(x) - f(x)$

Convexity of g, h

$$\begin{aligned} a_i &\leq a_{i+1} \\ a_i - m_i &\leq a_{i+1} - m_{i+1} \\ a_i x_i + b_i &= a_{i+1} x_i + b_{i+1} \end{aligned}$$

$$\begin{aligned} a_0 &= 0 \\ a_i &= a_{i-1} + \max\{0, m_i - m_{i-1}\} \\ b_0 &= 0 \\ b_i &= b_{i-1} + (a_{i-1} - a_i) x_{i-1} \end{aligned}$$

Obtaining dc decompositions

For $f \in \mathcal{C}^2(\Omega)$ and compact Ω,

We are done!!!

Take:

$$\begin{aligned}
\lambda^*(x) &: \text{lowest eigenvalue of } H_f(x)\text{, the Hessian of } f \text{ at } x \\
\tau &\geq \max\{0, -\frac{1}{2}\min_{x\in\Omega}\lambda^*(x)\} \\
f(x) &= \left(f(x) + \tau\|x\|^2\right) - \tau\|x\|^2
\end{aligned}$$

Obtaining dc decompositions

For $f \in \mathcal{C}^2(\Omega)$ and compact Ω,

We are done!!!

Take:

$$\begin{aligned} \lambda^*(x) &: \text{ lowest eigenvalue of } H_f(x), \text{ the Hessian of } f \text{ at } x \\ \tau &\geq \max\{0, -\frac{1}{2}\min_{x\in\Omega}\lambda^*(x)\} \\ f(x) &= \left(f(x) + \tau\|x\|^2\right) - \tau\|x\|^2 \end{aligned}$$

Obtaining dc decompositions

For $f \in \mathcal{C}^2(\Omega)$ and compact Ω,

Are we really done???

- $\min_{x \in \Omega} \lambda^*(x)$: to be (over)estimated
- Floudas α-BB:
  - Compute H, interval matrix, $H_f(x) \in H$
  - Compute bounds on eigenvalues of H

Obtaining dc decompositions

For $f \in \mathcal{C}^2(\Omega)$ and compact Ω,

Are we really done???

- $\min_{x \in \Omega} \lambda^*(x)$: to be (over)estimated
- Floudas α-BB:
  - Compute $\mathbf{H}$, interval matrix, $H_f(x) \in \mathbf{H}$
  - Compute bounds on eigenvalues of $\mathbf{H}$

Obtaining dc decompositions

For $f \in \mathcal{C}^2(\Omega)$ and compact Ω,

Are we really done???

- $\min_{x \in \Omega} \lambda^*(x)$: to be (over)estimated
- Floudas α-BB:
  - Compute $\mathbf{H}$, interval matrix, $H_f(x) \in \mathbf{H}$
  - Compute bounds on eigenvalues of $\mathbf{H}$

Properties of dc functions

Some algebra

$$\left.\begin{array}{rcl} f_1 & = & g_1 - h_1 \\ f_2 & = & g_2 - h_2 \end{array}\right\} \Rightarrow$$

- $-f_1 = h_1 - g_1$
- $(f_1 + f_2) = (g_1 + g_2) - (h_1 + h_2)$
- $\max\{f_1, f_2\} = \max\{g_1 - h_1, g_2 - h_2\} = \max\{g_1 + h_2, g_2 + h_1\} - (h_1 + h_2)$
- $\min\{f_1, f_2\} = \min\{g_1 - h_1, g_2 - h_2\} = g_1 + g_2 - \max\{g_2 + h_1, g_1 + h_2\}$
- $|f_1| = \max\{f_1, -f_1\} = 2\max\{g_1, h_1\} - (g_1 + h_1)$

Properties of dc functions

Some algebra

$$\left.\begin{array}{rcl} f_1 & = & g_1 - h_1 \\ f_2 & = & g_2 - h_2 \end{array}\right\} \Rightarrow$$

- $-f_1 = h_1 - g_1$
- $(f_1 + f_2) = (g_1 + g_2) - (h_1 + h_2)$
- $\max\{f_1, f_2\} = \max\{g_1 - h_1, g_2 - h_2\} = \max\{g_1 + h_2, g_2 + h_1\} - (h_1 + h_2)$
- $\min\{f_1, f_2\} = \min\{g_1 - h_1, g_2 - h_2\} = g_1 + g_2 - \max\{g_2 + h_1, g_1 + h_2\}$
- $|f_1| = \max\{f_1, -f_1\} = 2\max\{g_1, h_1\} - (g_1 + h_1)$

Properties of dc functions

Some algebra

$$\left.\begin{array}{rcl} f_1 & = & g_1 - h_1 \\ f_2 & = & g_2 - h_2 \end{array}\right\} \Rightarrow$$

- $-f_1 = h_1 - g_1$
- $(f_1 + f_2) = (g_1 + g_2) - (h_1 + h_2)$
- $\max\{f_1, f_2\} = \max\{g_1 - h_1, g_2 - h_2\} = \max\{g_1 + h_2, g_2 + h_1\} - (h_1 + h_2)$
- $\min\{f_1, f_2\} = \min\{g_1 - h_1, g_2 - h_2\} = g_1 + g_2 - \max\{g_2 + h_1, g_1 + h_2\}$
- $|f_1| = \max\{f_1, -f_1\} = 2\max\{g_1, h_1\} - (g_1 + h_1)$

Properties of dc functions

Some algebra

$$\left.\begin{array}{rcl} f_1 & = & g_1 - h_1 \\ f_2 & = & g_2 - h_2 \end{array}\right\} \Rightarrow$$

- $-f_1 = h_1 - g_1$
- $(f_1 + f_2) = (g_1 + g_2) - (h_1 + h_2)$
- $\max\{f_1, f_2\} = \max\{g_1 - h_1, g_2 - h_2\} = \max\{g_1 + h_2, g_2 + h_1\} - (h_1 + h_2)$
- $\min\{f_1, f_2\} = \min\{g_1 - h_1, g_2 - h_2\} = g_1 + g_2 - \max\{g_2 + h_1, g_1 + h_2\}$
- $|f_1| = \max\{f_1, -f_1\} = 2\max\{g_1, h_1\} - (g_1 + h_1)$

Properties of dc functions

Some algebra

$$\left.\begin{array}{lcl} f_1 & = & g_1 - h_1 \\ f_2 & = & g_2 - h_2 \end{array}\right\} \Rightarrow$$

- $-f_1 = h_1 - g_1$
- $(f_1 + f_2) = (g_1 + g_2) - (h_1 + h_2)$
- $\max\{f_1, f_2\} = \max\{g_1 - h_1, g_2 - h_2\} = \max\{g_1 + h_2, g_2 + h_1\} - (h_1 + h_2)$
- $\min\{f_1, f_2\} = \min\{g_1 - h_1, g_2 - h_2\} = g_1 + g_2 - \max\{g_2 + h_1, g_1 + h_2\}$
- $|f_1| = \max\{f_1, -f_1\} = 2\max\{g_1, h_1\} - (g_1 + h_1)$

Properties of dc functions

Some algebra

$$\left.\begin{array}{rcl} f_1 & = & g_1 - h_1 \\ f_2 & = & g_2 - h_2 \end{array}\right\} \Rightarrow$$

- $-f_1 = h_1 - g_1$
- $(f_1 + f_2) = (g_1 + g_2) - (h_1 + h_2)$
- $\max\{f_1, f_2\} = \max\{g_1 - h_1, g_2 - h_2\} = \max\{g_1 + h_2, g_2 + h_1\} - (h_1 + h_2)$
- $\min\{f_1, f_2\} = \min\{g_1 - h_1, g_2 - h_2\} = g_1 + g_2 - \max\{g_2 + h_1, g_1 + h_2\}$
- $|f_1| = \max\{f_1, -f_1\} = 2\max\{g_1, h_1\} - (g_1 + h_1)$

Composition

Let Ω be compact, let $f : \Omega \longrightarrow [0, a]$ be dc on Ω, with dc decomposition $f = g - h$. If $q : [0, a] \longrightarrow \mathbb{R}$ is convex nondecreasing with $q'_{-}(a) < \infty$, then $q \circ f$ is dc with dc decomposition

$$q(f(x)) = (q(f(x)) + Kh(x)) - Kh(x)$$

for any $K \geq q'_{-}(a)$.

Why?

- Define $\overline{q}(s, t) := q(s - t) + Kt$. $\overline{q}$: convex and componentwise nondecreasing.
- $g \circ f = \overline{q}(g, h) + Kh - Kh$

Composition

Let Ω be compact, let $f : \Omega \longrightarrow [0, a]$ be dc on Ω, with dc decomposition $f = g - h$. If $q : [0, a] \longrightarrow \mathbb{R}$ is convex nondecreasing with $q'_{-}(a) < \infty$, then $q \circ f$ is dc with dc decomposition

$$q(f(x)) = (q(f(x)) + Kh(x)) - Kh(x)$$

for any $K \geq q'_{-}(a)$.

Why?

- Define $\overline{q}(s, t) := q(s - t) + Kt$. $\overline{q}$: convex and componentwise nondecreasing.
- $g \circ f = \overline{q}(g, h) + Kh - Kh$

Composition

Let Ω be compact, let $f : \Omega \longrightarrow [0, a]$ be dc on Ω, with dc decomposition $f = g - h$. If $q : [0, a] \longrightarrow \mathbb{R}$ is convex nondecreasing with $q'_{-}(a) < \infty$, then $q \circ f$ is dc with dc decomposition

$$q(f(x)) = (q(f(x)) + Kh(x)) - Kh(x)$$

for any $K \geq q'_{-}(a)$.

Why?

- Define $\overline{q}(s, t) := q(s - t) + Kt$. $\overline{q}$: convex and componentwise nondecreasing.
- $g \circ f = \overline{q}(g, h) + Kh - Kh$

Composition

Let Ω be a compact convex set. $f : \Omega \longrightarrow \mathbb{R}_+$ be dc on Ω, with dc decomposition $f = g - h$. If $q : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is convex nonincreasing with $q'_+(0) > -\infty$, then $q \circ f$ is dc with dc decomposition

$$q(f(x)) = (q(f(x)) + K(g(x) + h(x))) - K(g(x) + h(x))$$

for any $K \geq q'_+(0)$.

Composition

Let Ω be a compact convex set. $f : \Omega \longrightarrow \mathbb{R}_+$ be dc on Ω, with dc decomposition $f = g - h$. If $q : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is concave nondecreasing with $q'_+(0) < \infty$, then $q \circ f$ is dc with dc decomposition

$$q(f(x)) = K(g(x) + h(x)) - (K(g(x) + h(x)) - q(f(x)))$$

for any $K \geq q'_+(0)$.

Norm of a dc function

Let $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^k$ be componentwise dc:
$(f_1, \ldots, f_k) = (g_1, \ldots, g_k) - (h_1, \ldots, h_k)$. Let $\|\cdot\|$ be a norm in $\mathbb{R}^k$. Then,

$$\|(f_1, \ldots, f_k)\| = \|(f_1, \ldots, f_k)\| + \sum^k \|e_i\|(g_i + h_i) - \sum^k \|e_i\|(g_i + h_i)$$

Why?

$$\begin{aligned} \|(f_1, \ldots, f_k)\| &= \max_{\|u\|^\circ \leq 1} \sum_{i=1}^k u_i\,(g_i - h_i) \\ &= \max_{\|u\|^\circ \leq 1} \sum_{i=1}^k ((u_i + M_i)g_i + (M_i - u_i)h_i) - \sum_{i=1}^k M_i(g_i + h_i) \end{aligned}$$

$\|f\| = \max_{\|u\|^\circ \leq 1} \sum_{i=1}^k ((u_i + M_i)g_i + (M_i - u_i)h_i) - \sum_{i=1}^k M_i(g_i + h_i)$

Norm of a dc function

Let $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^k$ be componentwise dc:
$(f_1, \ldots, f_k) = (g_1, \ldots, g_k) - (h_1, \ldots, h_k)$. Let $\| \cdot \|$ be a norm in $\mathbb{R}^k$. Then,

$$\|(f_1, \ldots, f_k)\| = \|(f_1, \ldots, f_k)\| + \sum^k \|e_i\|(g_i + h_i) - \sum^k \|e_i\|(g_i + h_i)$$

Why?

$$\begin{aligned}
\|(f_1, \ldots, f_k)\| &= \max_{\|u\|^\circ \leq 1} \sum_{i=1}^k u_i\,(g_i - h_i) \\
&= \max_{\|u\|^\circ \leq 1} \sum_{i=1}^k ((u_i + M_i)g_i + (M_i - u_i)h_i) - \sum_{i=1}^k M_i(g_i + h_i)
\end{aligned}$$

$\|f\| = \max_{\|u\|^\circ \leq 1} \sum_{i=1}^k ((u_i + M_i)g_i + (M_i - u_i)h_i) - \sum_{i=1}^k M_i(g_i + h_i)$

Norm of a dc function

Let $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^k_+$ be componentwise dc: $(f_1, \ldots, f_k) = (g_1, \ldots, g_k) - (h_1, \ldots, h_k)$. Let $\| \cdot \|$ be a norm monotonic in $\mathbb{R}^k_+$.

Then, $\|f\|$ is dc on Ω, with dc decomposition

$$\|(f_1, \ldots, f_k)\| = \|(f_1, \ldots, f_k)\| + \sum_{i=1}^{k} \|e_i\| h_i - \sum_{i=1}^{k} \|e_i\| h_i$$

Ordered functions

- Let $s_{(r)}$: the r-th smallest value in $s_1, \ldots, s_k$.
- Let $\|(s_1, \ldots, s_k)\|_r = \sum_{i=r}^k |s|_{(i)}$. $\| \cdot \|_r$: norm, monotonic in $\mathbb{R}^k_+$
- For $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^k_+$ componentwise dc:
  $(f_1, \ldots, f_k) = (g_1, \ldots, g_k) - (h_1, \ldots, h_k)$,

$$\|f\|_{(r)} = \|f\|_{(r)} + \sum_{i=1}^k h_i - \sum_{i=1}^k h_i, \|f\|_{(r+1)} = \|f\|_{(r+1)} + \sum_{i=1}^k h_i - \sum_{i=1}^k h_i$$

$$f_{(r)} = \|f\|_{(r)} + \sum_{i=1}^k h_i - \left(\|f\|_{(r+1)} + \sum_{i=1}^k h_i \right)$$

Does this small thing matter?

$$\min_{(x,y)\in\mathbb{R}^2:0\leq x\leq 6,0\leq y\leq 4} \left\| \left(\frac{\overline{f}_i - f_i(x)}{\overline{f}_i - \underline{f}_i} \right)_i \right\|$$

$f_1(x,y) = x + 10\cos(5\pi y)$ $\overline{f}_1 = 16$ $\underline{f}_1 = -10$

$f_2(x,y) = -x^2 + 6x + y^2$ $\overline{f}_2 = 25$ $\underline{f}_2 = 0$

$f_3(x,y) = 10x + 7y$ $\overline{f}_3 = 88$ $\underline{f}_3 = 0$

$f_4(x,y) = -x^2 + 2x + 4y$ $\overline{f}_4 = 17$ $\underline{f}_4 = -0$

$f_5(x,y) = x^2 + y^2$ $\overline{f}_5 = 52$ $\underline{f}_5 = 0$

Does this small thing matter?

Indeed ...

	Iterations			Vertices			
dec	Min	Max	Ave	Min	Max	Ave	Time (s)
1	2608	8220	7568	5218	16442	15139	9.500
2	1303	2913	2809	2608	5828	5621	3.078

Distance to a set

Given $S \subset \mathbb{R}^n$, define $d_S(x) = \inf\{\|y - x\| : y \in S\}$

- d_S^2 is dc, with dc decomposition
  $d_S^2(x) = \|x\|^2 - (\|x\|^2 - d_S^2(x))$

Why?

$$\begin{aligned} d_S^2(x) &= \inf\{\|y - x\|^2 : y \in S\} \\ &= \|x\|^2 + \inf\{\|y\|^2 - 2x^\top y : y \in S\} \\ &= \|x\|^2 - (\|x\|^2 - d_S^2(x)) \end{aligned}$$

Distance to a set

Given $S \subset \mathbb{R}^n$, define $d_S(x) = \inf\{\|y - x\| : y \in S\}$

- d_S^2 is dc, with dc decomposition
  $d_S^2(x) = \|x\|^2 - (\|x\|^2 - d_S^2(x))$

Why?

$$
\begin{aligned}
d_S^2(x) &= \inf\{\|y - x\|^2 : y \in S\} \\
&= \|x\|^2 + \inf\{\|y\|^2 - 2x^\top y : y \in S\} \\
&= \|x\|^2 - (\|x\|^2 - d_S^2(x))
\end{aligned}
$$

Distance to a set

However, d_S may not be dc ...

Take $S = \{1/2^n\}_{n \in \mathbb{N}} \cup \{0\}$. The right-derivative of d_S at 0 does not exist, so d_S cannot be written as the difference of two convex functions

... though it is in some interesting cases

Let $S \subset \mathbb{R}^k$ be convex and $f : \mathbb{R}^n \longmapsto \mathbb{R}^k$ componentwise dc, with $f_j = g_j - h_j$, $j = 1, \ldots k$. Then, $d_S \circ f = \varphi^+ - \varphi^-$ with

$$\varphi^+ = d_S \circ f + \sum_{j=1}^{k} (g_j + h_j)$$

$$\varphi^- = \sum_{j=1}^{k} (g_j + h_j).$$

Similar result for S : complement of convex.

For $S = \bigcup_{i=1}^{r} S_i$, with convex/complement of convex S_i,
$d_S = \min_{1 \leq i \leq r} d_{S_i} \ldots$

http://www.springer.com/11750

Fields of application

- Machine Learning, e.g. Shen et al. J. Amer. Statist. Assoc. (2003).
- Finance, e.g. Konno et al. Comput. Optim. Appl. (2005).
- Statistics, e.g. Blanquero et al. JOGO (2001).
- Computational Chemistry, e.g. An and Tao, SIAM J. Optim. (2003).
- Logistics, e.g. Holmberg and Tuy, Math. Program. (1999).
- Many more fields: An and Tao, Ann. Oper. Res. (2005).

Applications in Continuous Locational Analysis

Where to locate one or several facilities in the plane to optimize a certain performance measure of the distance users/facilities

Applications in Continuous Locational Analysis

Where to locate one or several facilities in the plane to optimize a certain performance measure of the distance users/facilities

Applications in Continuous Locational Analysis

Where to locate one or several facilities in the plane to optimize a certain performance measure of the distance users/facilities

Applications in Continuous Locational Analysis

Where to locate one or several facilities in the plane to optimize a certain performance measure of the distance users/facilities

$$\min_{x \in S} \sum_{a \in A} \varphi_a(\|x - a\|),$$

- φ_a : dc, with dc decomposition easy to obtain.
- S : polygone in $\mathbb{R}^2$.

$$\min_{x \in S} \sum_{a \in A} \varphi_a(\|x - a\|),$$

- φ_a : dc, with dc decomposition easy to obtain.
- S : polygone in $\mathbb{R}^2$.

Obnoxious facility location

$$\min_x \quad \sum_{a \in A} \frac{\omega_a}{\|x - a\|^2} \qquad (\omega_a > 0 \quad \forall a)$$

DC decomposition: $\varphi_a^1(d) = \omega_a / d^2$ $\qquad \varphi_a^2(d) = 0$

Weber problem with some negative weights

$$\min_x \quad \sum_{a \in A} \omega_a \|x - a\| \qquad (\omega_a \in \mathbb{R})$$

DC decomposition :

$$\varphi_a^1(d) = \max\{\omega_a, 0\} d \qquad \varphi_a^2(d) = \max\{-\omega_a, 0\} d$$

Obnoxious facility location

$$\min_{x} \quad \sum_{a \in A} \frac{\omega_a}{\|x-a\|^2} \qquad (\omega_a > 0 \quad \forall a)$$

DC decomposition: $\varphi_a^1(d) = \omega_a / d^2$ $\qquad \varphi_a^2(d) = 0$

Weber problem with some negative weights

$$\min_{x} \quad \sum_{a \in A} \omega_a \|x-a\| \qquad (\omega_a \in \mathbb{R})$$

DC decomposition :

$$\varphi_a^1(d) = \max\{\omega_a, 0\} d \qquad \varphi_a^2(d) = \max\{-\omega_a, 0\} d$$

Maximal capture

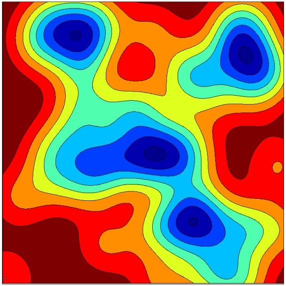
$$\max_x \sum_{a \in A} \omega_a \exp(-\|x - a\|)$$

DC decomposition $\varphi_a^1(d) = 0 \qquad \varphi_a^2(d) = \omega_a \exp(-d)$

Maximal capture

$$\max_{x} \quad \sum_{a \in A} \omega_a \exp(-\|x - a\|)$$

DC decomposition $\quad \varphi_a^1(d) = 0 \qquad \varphi_a^2(d) = \omega_a \exp(-d)$



Huff competitive location

$$\max_x \quad \sum_{a \in A} \frac{\omega_a}{1 + h_a \|x - a\|^\lambda} \qquad (h_a, \omega_a > 0 \quad \lambda \geq 1)$$

Huff competitive location

$$\max_{x} \quad \sum_{a \in A} \frac{\omega_a}{1 + h_a \|x - a\|^{\lambda}} \qquad (h_a, \omega_a > 0 \quad \lambda \geq 1)$$

DC decomposition

$$\begin{aligned} \varphi_a^1(d) &= \omega_a h_a d^{\lambda} \\ \varphi_a^2(d) &= \phi_a(d) + \omega_a h_a d^{\lambda} \\ \phi_a(d) &= \frac{\omega_a}{1 + h_a d^{\lambda}} \end{aligned}$$

Huff competitive location

$$\max_x \sum_{a \in A} \frac{\omega_a}{1 + h_a \|x - a\|^\lambda} \qquad (h_a, \omega_a > 0 \quad \lambda \geq 1)$$

Another DC decomposition

$$\varphi_a^1(d) = \begin{cases} \phi_a(\bar{d}) + \phi_a'(\bar{d})(d - \bar{d}) - \phi_a(d) & \text{si } d < \bar{d} \\ 0 & \text{si } d \geq \bar{d} \end{cases}$$

$$\varphi_a^2(d) = \begin{cases} \phi_a(\bar{d}) + \phi_a'(\bar{d})(d - \bar{d}) & \text{si } d < \bar{d} \\ \phi_a(d) & \text{si } d \geq \bar{d} \end{cases}$$

$\bar{d}$: root of equation $\phi_a''(d) = 0$.

Huff model with uncertainty in the demand

T. Drezner (2009), Bello-Blanquero-Carrizosa (2009)

- A set E of scenarios; for each scenario $e \in E$,
  - a vector $(\omega_a^e)_{a \in A}$ of weights is given.
  - ideal market share for the entering firm: $z^e = \max_{x \in S} \sum_{a \in A} \omega_a^e \frac{1}{1+\beta_a \|a-x\|^\lambda}$.
- Minimax regret location:

$$\min_{x \in S} \left\| \left(z^e - \sum_{a \in A} \omega_a^e \frac{1}{1 + \beta_a \|a - x\|^\lambda} \right)_{e \in E} \right\|_*,$$

$\| \cdot \|_*$: norm

Huff model with uncertainty in the demand

T. Drezner (2009), Bello-Blanquero-Carrizosa (2009)

- A set E of scenarios; for each scenario $e \in E$,
  - a vector $(\omega_a^e)_{a\in A}$ of weights is given.
  - ideal market share for the entering firm: $z^e = \max_{x\in S} \sum_{a\in A} \omega_a^e \frac{1}{1+\beta_a \|a-x\|^\lambda}$.
- Minimax regret location:

$$\min_{x\in S} \left\| \left(z^e - \sum_{a\in A} \omega_a^e \frac{1}{1+\beta_a \|a-x\|^\lambda} \right)_{e\in E} \right\|_* ,$$

$\|\cdot\|_*$: norm

Huff model with uncertainty in the demand

$$\min_{x \in S} \left\| \left(z^e - \sum_{a \in A} \omega_a^e \frac{1}{1 + \beta_a \|a - x\|^\lambda} \right)_{e \in E} \right\|_*$$

If $\| \cdot \|_*$: monotonic in the positive orthant, then the function $d := (d_a)_{a \in A} \longmapsto \left\| \left(z^e - \sum_{a \in A} \omega_a^e \frac{1}{1+\beta_a d_a^\lambda} \right)_{e \in E} \right\|_*$: difference of two convex componentwise nonincreasing functions.

Robust Huff location model

Robustness measure: Carrizosa-Nickel, 2003

- We are given a Huff model with
  - a nominal value $\widehat{\omega}$ for the weight vector
  - a threshold value τ
- The robustness $\rho(x)$ of solution x is measured as the maximal possible deviation in the weight vector w.r.t. its nominal value $\widehat{\omega}$ so that the total market capture remains above the threshold τ.
- Aim: $\max_{x \in S} \rho(x)$

Robust Huff location model

Robustness measure: Carrizosa-Nickel, 2003

- We are given a Huff model with
  - a nominal value $\widehat{\omega}$ for the weight vector
  - a threshold value τ
- The robustness $\rho(x)$ of solution x is measured as the maximal possible deviation in the weight vector w.r.t. its nominal value $\widehat{\omega}$ so that the total market capture remains above the threshold τ.
- Aim: $\max_{x \in S} \rho(x)$

Robust Huff location model

Robustness measure: Carrizosa-Nickel, 2003

- We are given a Huff model with
  - a nominal value $\widehat{\omega}$ for the weight vector
  - a threshold value τ
- The robustness $\rho(x)$ of solution x is measured as the maximal possible deviation in the weight vector w.r.t. its nominal value $\widehat{\omega}$ so that the total market capture remains above the threshold τ.
- Aim: $\max_{x \in S} \rho(x)$

Robust Huff location model

$$\rho(x) = \max\left\{0, \frac{\sum_{a\in A} \frac{\widehat{\omega}_a}{1+\beta_a\|x-a\|^{\lambda}} - \tau}{\|(\|x-a\|_{a\in A})\|_*}\right\}$$

Particular case: $\|u\|_* = \sqrt{u^\top \Sigma u}$

$\max_x \rho(x)$

$$\max_x P\left(\sum_{a\in A} \frac{\omega_a}{1+\beta_a\|x-a\|^{\lambda}} \geq \tau\right)$$

$$(\omega \sim \mathcal{N}(\widehat{\omega}, \Sigma))$$

Robust Huff location model

$$\rho(x) = \max\left\{0, \frac{\sum_{a\in A} \frac{\widehat{\omega_a}}{1+\beta_a\|x-a\|^\lambda} - \tau}{\|(\|x-a\|_{a\in A})\|_*}\right\}$$

Particular case: $\|u\|_* = \sqrt{u^\top \Sigma u}$

$$max_x \rho(x)$$

$$\max_x P\left(\sum_{a\in A} \frac{\omega_a}{1+\beta_a\|x-a\|^\lambda} \geq \tau\right)$$

$$(\omega \sim \mathcal{N}(\widehat{\omega}, \Sigma))$$

Robust Huff location model

$$\rho(x) = \max\left\{0, \frac{\sum_{a\in A} \frac{\widehat{\omega_a}}{1+\beta_a\|x-a\|^\lambda} - \tau}{\|(\|x-a\|_{a\in A})\|_*}\right\}$$

Particular case: $\|u\|_* = \sqrt{u^\top \Sigma u}$

$max_x \rho(x)$

$$\max_x P\left(\sum_{a\in A} \frac{\omega_a}{1+\beta_a\|x-a\|^\lambda} \geq \tau\right)$$

$$(\omega \sim \mathcal{N}(\widehat{\omega}, \Sigma))$$

Stochastic weighted minimax

$$\max_x \quad \sum_{a \in A} \ln(\|x-a\|) - \ln(T_a - \alpha_a \|x-a\|)$$

DC decomposition $\varphi_a^1(d) = -\ln(d) \quad \varphi_a^2(d) = -\ln(T_a - \alpha_a d)$

Inventory-location model

$$\min_x \quad \sum_{a \in A} \alpha_a \|x-a\| + \omega_a \sqrt{R_a \|x-a\|^2 + S_a \|x-a\| + T_a}$$

DC decomposition

$$\varphi_a^1(d) = 0 \qquad \varphi_a^2(d) = -\alpha_a d - \omega \sqrt{R_a d^2 + S_a d + T_a}$$

Stochastic weighted minimax

$$\max_{x} \quad \sum_{a\in A} \ln(\|x-a\|) - \ln(T_a - \alpha_a\|x-a\|)$$

DC decomposition $\varphi_a^1(d) = -\ln(d) \quad \varphi_a^2(d) = -\ln(T_a - \alpha_a d)$

Inventory-location model

$$\min_{x} \quad \sum_{a\in A} \alpha_a\|x-a\| + \omega_a\sqrt{R_a\|x-a\|^2 + S_a\|x-a\| + T_a}$$

DC decomposition

$$\varphi_a^1(d) = 0 \qquad \varphi_a^2(d) = -\alpha_a d - \omega\sqrt{R_a d^2 + S_a d + T_a}$$

Gradual covering

$$\min_{x} \quad \sum_{a \in A} \phi_a(x) := \begin{cases} 0 & \text{if } \|x-a\| \leq l_a \\ \omega_a(\|x-a\| - l_a) & \text{if } l_a < \|x-a\| \leq u_a \\ \omega_a(u_a - l_a) & \text{if } \|x-a\| > u_a \end{cases}$$

DC decomposition

$$\varphi_a^1(d) = \begin{cases} 0 & \text{if } d < l_a \\ \omega(d - l_a) & \text{if } d \geq l_a \end{cases}$$

$$\varphi_a^2(d) = \begin{cases} 0 & \text{if } d < u_a \\ \omega_a(d - u_a) & \text{if } d \geq u_a \end{cases}$$

Gradual covering

$$\min_x \quad \sum_{a \in A} \phi_a(x) := \begin{cases} 0 & \text{if } \|x-a\| \leq l_a \\ \omega_a(\|x-a\| - l_a) & \text{if } l_a < \|x-a\| \leq u_a \\ \omega_a(u_a - l_a) & \text{if } \|x-a\| > u_a \end{cases}$$

DC decomposition

$$\varphi_a^1(d) = \begin{cases} 0 & \text{if } d < l_a \\ \omega(d - l_a) & \text{if } d \geq l_a \end{cases}$$

$$\varphi_a^2(d) = \begin{cases} 0 & \text{if } d < u_a \\ \omega_a(d - u_a) & \text{if } d \geq u_a \end{cases}$$

The acceleration-deceleration distance

$$\min_x \quad \sum_{a \in A} \phi_a(x)$$

donde

$$\phi_a(x) = \begin{cases} 2\sqrt{\|x-a\| d_{0a}} & \text{if } \|x-a\| < d_{0a} \\ \|x-a\| + d_{0a} & \text{if } \|x-a\| \geq d_{0a} \end{cases}$$

DC decomposition

$$\varphi_a^1(d) = 0 \qquad \varphi_a^2(d) = -\begin{cases} 2\sqrt{d d_{0a}} & \text{if } d < d_{0a} \\ d + d_{0a} & \text{if } d \geq d_{0a} \end{cases}$$

The acceleration-deceleration distance

$$\min_{x} \sum_{a \in A} \phi_a(x)$$

donde

$$\phi_a(x) = \begin{cases} 2\sqrt{\|x-a\| d_{0a}} & \text{if } \|x-a\| < d_{0a} \\ \|x-a\| + d_{0a} & \text{if } \|x-a\| \geq d_{0a} \end{cases}$$

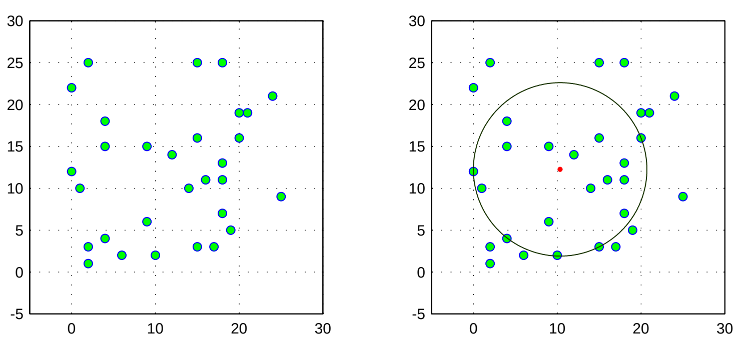
DC decomposition

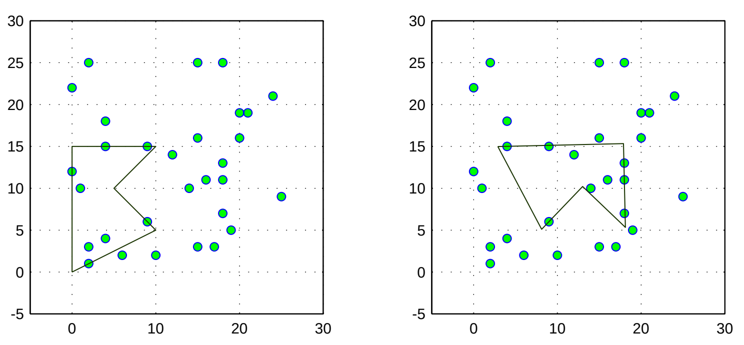
$$\varphi_a^1(d) = 0 \qquad \varphi_a^2(d) = -\begin{cases} 2\sqrt{d d_{0a}} & \text{if } d < d_{0a} \\ d + d_{0a} & \text{if } d \geq d_{0a} \end{cases}$$

Location of structures

- A : set of points
- S : structure, fixed (e.g., a polygone)
- The translation and rotation τ of S is sought so as to minimize a certain function $f(d_S(\tau(a))_{a \in A})$:

$$\min_{\vartheta, u} f\left(d_S\left(u + \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix} a\right)_{a \in A}\right)$$


30
25
20
15
10
5
0
-5
0
10
20
30
30
25
20
15
10
5
0
-5
0
10
20
30


30
25
20
15
10
5
0
-5
0
10
20
30

Biobjective location problems

$$\begin{cases} \min & \sum_{a\in A_+} \omega_a \|x-a\| \\ \max & \min_{1\le i\le r} d_{R_i}(x) \\ & x\in S \end{cases}$$

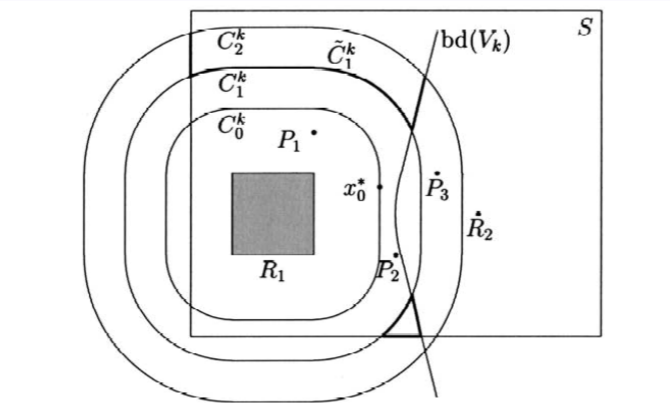
$$\begin{array}{ll} \min & \sum_{a\in A_+} \omega_a \|x-a\| \\ & d_{R_i}(x) \ge \varepsilon\, \forall i \\ & x\in S \end{array}$$

Biobjective location problems

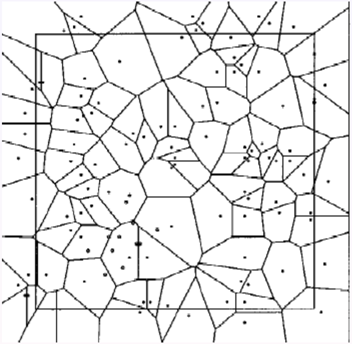
$$\begin{cases} \min & \sum_{a \in A_+} \omega_a \|x - a\| \\ \max & \min_{1 \leq i \leq r} d_{R_i}(x) \\ & x \in S \end{cases}$$

$$\begin{array}{ll} \min & \sum_{a \in A_+} \omega_a \|x - a\| \\ & d_{R_i}(x) \geq \varepsilon \, \forall i \\ & x \in S \end{array}$$

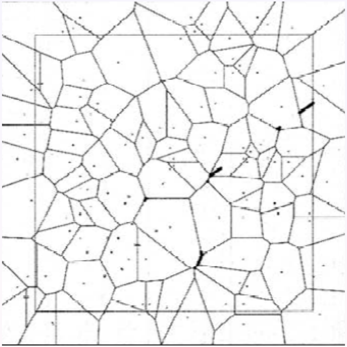
Biobjective location problems



Biobjective location problems



Biobjective location problems



Algorithms (Hansen-Jaumard-Tuy, 1995)

- Branch and bound
  - Branching:
    - Big Square Small Square (Hansen-Peeters-Richard-Thisse, 1985)
    - Big Triangle Small Triangle (Drezner-Suzuki, 2004)
  - Bounding: exploiting
    - monotonicity
    - dc (Drezner, 2007)
- Outer approximation
- . . .

Algorithms (Hansen-Jaumard-Tuy, 1995)

- Branch and bound
  - Branching:
    - Big Square Small Square (Hansen-Peeters-Richard-Thisse, 1985)
    - Big Triangle Small Triangle (Drezner-Suzuki, 2004)
  - Bounding: exploiting
    - monotonicity
    - dc (Drezner, 2007)
- Outer approximation
- ...

Algorithms (Hansen-Jaumard-Tuy, 1995)

- Branch and bound
  - Branching:
    - Big Square Small Square (Hansen-Peeters-Richard-Thisse, 1985)
    - Big Triangle Small Triangle (Drezner-Suzuki, 2004)
  - Bounding: exploiting
    - monotonicity
    - dc (Drezner, 2007)
- Outer approximation
- ...

$$\min_{x \in S} f(x)$$

- f : dc, with dc decomposition $f = g - h$ available
- S : polytope in $\mathbb{R}^n$

Aim

to find an ε-optimal solution

Concave minimization

$$\min_{x \in S} f(x)$$

- Suppose f : concave.
- An optimal solution of f : at some point in $ext(S)$.
- Hard in general. Simple, for instance, if S : simplex or low-dimensional.

Concave minimization

$$\min_{x \in S} f(x)$$

- Suppose f : concave.
- An optimal solution of f : at some point in $ext(S)$.
- Hard in general. Simple, for instance, if S : simplex or low-dimensional.

Concave minimization

$$\min_{x \in S} f(x)$$

- Suppose f : concave.
- An optimal solution of f : at some point in $ext(S)$.
- Hard in general. Simple, for instance, if S : simplex or low-dimensional.

Constructing concave lower envelopes

- $f = g - h$
- Take $x^* \in S \cap$ rint dom(g), and take $p \in \partial g(x^*)$
- By definition of subgradient, $g(x) \geq g(x^*) + p^\top (x - x^*) \forall x$
- $f(x) \geq g(x^*) + p^\top (x - x^*) - h(x)$, a concave function

Constructing concave lower envelopes

- $f = g - h$
- Take $x^* \in S \cap$ rint dom(g), and take $p \in \partial g(x^*)$
- By definition of subgradient, $g(x) \geq g(x^*) + p^\top (x - x^*) \forall x$
- $f(x) \geq g(x^*) + p^\top (x - x^*) - h(x)$, a concave function

Constructing concave lower envelopes

- $f = g - h$
- Take $x^* \in S \cap \text{rint dom(g)}$, and take $p \in \partial g(x^*)$
- By definition of subgradient, $g(x) \geq g(x^*) + p^\top(x - x^*) \, \forall x$
- $f(x) \geq g(x^*) + p^\top(x - x^*) - h(x)$, a concave function

Constructing concave lower envelopes

- $f = g - h$
- Take $x^* \in S \cap \text{rint dom(g)}$, and take $p \in \partial g(x^*)$
- By definition of subgradient, $g(x) \geq g(x^*) + p^\top (x - x^*) \, \forall x$
- $f(x) \geq g(x^*) + p^\top (x - x^*) - h(x)$, a concave function

Constructing concave lower envelopes

- $f = g - h$
- Take $x^* \in S \cap \text{rint dom(g)}$, and take $p \in \partial g(x^*)$
- By definition of subgradient, $g(x) \geq g(x^*) + p^\top (x - x^*) \, \forall x$
- $f(x) \geq g(x^*) + p^\top (x - x^*) - h(x)$, a concave function

Degree of freedom: choice of the dc decomposition

- $f = g + k - (h + k)$
- Take $x^* \in S \cap \text{rint dom(g)} \cap \text{rint dom(k)}$, and take $p \in \partial g(x^*)$, $q \in \partial k(x^*)$,
- $f(x) \geq g(x^*) + p^\top (x - x^*) - h(x) + k(x^*) + q^\top (x - x^*) - k(x)$, a concave function

Constructing concave lower envelopes

- $f = g - h$
- Take $x^* \in S \cap \text{rint dom(g)}$, and take $p \in \partial g(x^*)$
- By definition of subgradient, $g(x) \geq g(x^*) + p^\top (x - x^*) \, \forall x$
- $f(x) \geq g(x^*) + p^\top (x - x^*) - h(x)$, a concave function

Degree of freedom: choice of the point x^*

$$z^* := \max_{x^* \in S, p \in \partial g(x^*)} \min_{x \in ext(S)} g(x^*) + p^\top (x - x^*) - h(x)$$

For S : simplex ...

$$z^* = \min_{x \in S} \left(g(x) - u^\top x\right) - \left(\min_{x \in ext(S)} h(x) - u^\top x\right)$$

$$u \; : \; h(x) - u^\top x \text{ constant in } ext(S)$$

Constructing concave lower envelopes

- $f = g - h$
- Take $x^* \in S \cap \text{rint dom(g)}$, and take $p \in \partial g(x^*)$
- By definition of subgradient, $g(x) \geq g(x^*) + p^\top (x - x^*) \, \forall x$
- $f(x) \geq g(x^*) + p^\top (x - x^*) - h(x)$, a concave function

Degree of freedom: choice of the point x^*

$$z^* := \max_{x^* \in S, p \in \partial g(x^*)} \min_{x \in ext(S)} g(x^*) + p^\top (x - x^*) - h(x)$$

For S : simplex . . .

$$z^* = \min_{x \in S} \left(g(x) - u^\top x\right) - \left(\min_{x \in ext(S)} h(x) - u^\top x\right)$$

$$u \quad : \quad h(x) - u^\top x \text{ constant in } ext(S)$$

Covering algorithm

Basic idea

Build a feasible sequence $\{x_k\}$ and a sequence of lower envelopes $\{E^{(k)}\}_k$, i.e., functions satisfying

$$\begin{array}{lcll} E^{(k)} & \leq & f & \text{for all } k \\ E^{(k)} & \leq & E^{(k+1)} & \text{for all } k \\ E^{(k)}(x_j) & = & f(x_j) & \text{for all } j = 1, \ldots, k, \\ x_{k+1} & \in & \arg\min_{x \in S} E^{(k)}(x) & \end{array}$$

Covering algorithm

Initialization

Take $x_1 \in S$

Set $\overline{f} = f(x_1)$, $x_* = x_1$ and construct $E^{(1)}$

Iteration $k = 1, 2, \ldots$

Set $\underline{f} = \min_{x \in S} E^{(k)}(x)$ and select $x_{k+1} \in \arg\min_{x \in S} E^{(k)}(x)$.

If $\overline{f} - \underline{f} \leq \varepsilon$ then Stop.

Construct $E^{(k+1)}$ from $E^{(k)}$.

Update, if needed, $\overline{f}$ and x_*

GoTo iteration $k+1$.

Where's the problem?

$$\min_{x \in S} E^{(k)}(x)$$

$$E^{(k)}(x) = \max_{1 \leq i \leq k} \left(g(x_i) + p_i^\top (x - x_i) - h(x) \right)$$

- Optimizing $E^{(k)}$, itself a Global-Optimization (in general non-differentiable) problem
- The methods of Breiman-Cutler (Math Prog, 1993), and Baritompa-Cutler (JOGO, 1994), shown to be particular cases by Blanquero-Carrizosa (JOGO, 2000).

Where's the problem?

$$\min_{x \in S} E^{(k)}(x)$$

$$E^{(k)}(x) = \max_{1 \leq i \leq k} \left(g(x_i) + p_i^\top (x - x_i) - h(x) \right)$$

- Optimizing $E^{(k)}$, itself a Global-Optimization (in general non-differentiable) problem
- The methods of Breiman-Cutler (Math Prog, 1993), and Baritompa-Cutler (JOGO, 1994), shown to be particular cases by Blanquero-Carrizosa (JOGO, 2000).

$$\min_{x\in S} E^{(k)}(x) := \max_{1\le i\le k} \left(g(x_i) + p_i^\top (x - x_i) - h(x) \right)$$

$$\min_{x\in S} E^{(k)}(x) := \max_{1\le i\le k} \left(g(x_i) + p_i^\top (x - x_i) \right) - h(x)$$

$$\begin{array}{lll} \min & t - h(x) & \\ \text{s.t.} & t \ge \left(g(x_i) + p_i^\top (x - x_i) \right) & i = 1, 2, \ldots, k \\ & x \in S & \end{array}$$

$$\min_{x\in S} E^{(k)}(x) := \max_{1\leq i\leq k} \Big(g(x_i) + p_i^\top (x - x_i) - h(x) \Big)$$

$$\min_{x\in S} E^{(k)}(x) := \max_{1\leq i\leq k} \Big(g(x_i) + p_i^\top (x - x_i) \Big) - h(x)$$

$$\begin{array}{lll} \min & t - h(x) & \\ \text{s.t.} & t \geq (g(x_i) + p_i^\top (x - x_i)) & i = 1, 2, \ldots, k \\ & x \in S & \end{array}$$

$$\min_{x\in S} E^{(k)}(x) := \max_{1\le i\le k} \Big(g(x_i) + p_i^\top (x - x_i) - h(x) \Big)$$

$$\min_{x\in S} E^{(k)}(x) := \max_{1\le i\le k} \Big(g(x_i) + p_i^\top (x - x_i) \Big) - h(x)$$

$$\begin{array}{lll} \min & t - h(x) & \\ \text{s.t.} & t \ge \big(g(x_i) + p_i^\top (x - x_i) \big) & i = 1, 2, \ldots, k \\ & x \in S & \end{array}$$

$$
\begin{array}{lll}
\min & t - h(x) & \\
\text{s.t.} & t \geq \left(g(x_i) + p_i^\top (x - x_i)\right) & i = 1, 2, \ldots, k \\
& x \in S &
\end{array}
$$

- Concave minimization on a polyhedral region (vertex inspection).
- At each stage, one cut added. On-line vertex enumeration procedures, e.g. Chen et al., Opns Res Letters (1991).

$$
\begin{array}{lll}
\min & t - h(x) & \\
\text{s.t.} & t \geq \left(g(x_i) + p_i^\top (x - x_i)\right) & i = 1, 2, \ldots, k \\
& x \in S &
\end{array}
$$

- Concave minimization on a polyhedral region (vertex inspection).
- At each stage, one cut added. On-line vertex enumeration procedures, e.g. Chen et al., Opns Res Letters (1991).

Does the dc decomposition chosen affect?

$f : [-0.2, 1] \subset \mathbb{R} \longmapsto \mathbb{R},\ f(x) = \cos(5\pi x) - x^2$

- Baritompa-Cutler (1994):

$$\begin{aligned} f_1(x) &= Kx^2 \\ f_2(x) &= (Kx^2 - f(x)), \end{aligned}$$

  $K \geq 12.5\pi^2 - 1.$

- Blanquero-Carrizosa (2000):

$$\begin{aligned} f_1(x) &= f(\hat{x}) + f'(\hat{x})(x - \hat{x}) + \int_{\hat{x}}^{x} (x - t)[f''(t)]^+ dt \\ f_2(x) &= \int_{\hat{x}}^{x} (x - t)[f''(t)]^- dt \end{aligned}$$

$f(x) = (f(x) + h(x)) - h(x),$

$$h(x) = \begin{cases} h_1(x) & x \in [z_1, a_1] \\ h_2(x) & x \in [a_1, b_1] \\ h_3(x) & x \in [b_1, a_2] \\ h_4(x) & x \in [a_2, b_2] \\ h_5(x) & x \in [b_2, a_3] \\ h_6(x) & x \in [a_3, b_3] \\ h_7(x) & x \in [b_3, z_2] \end{cases}$$

$f(x) = (f(x) + h(x)) - h(x)$,

$$
\begin{aligned}
h_1(x) &= 1 + 5\pi(x - a_1)\sin(5\pi a_1) - \cos(5\pi a_1) + a_1(2x - a_1)\\
h_2(x) &= 1 - \cos(5\pi x) + x^2\\
h_3(x) &= 1 + 5\pi(x - b_1)\sin(5\pi b_1) - \cos(5\pi b_1) + b_1(2x - b_1)\\
h_4(x) &= h_2(x) + h_3(x) - 1 - 5\pi(x - a_2)\sin(5\pi a_2) + \cos(5\pi a_2) - a_2(2x - a_2)\\
h_5(x) &= 1 + h_4(x) - h_2(x) + 5\pi(x - b_2)\sin(5\pi b_2) - \cos(5\pi b_2) + b_2(2x - b_2)\\
h_6(x) &= h_2(x) + h_5(x) - 1 - 5\pi(x - a_3)\sin(5\pi a_3) + \cos(5\pi a_3) - a_3(2x - a_3)\\
h_7(x) &= 1 + h_6(x) - h_2(x) + 5\pi(x - b_3)\sin(5\pi b_3) - \cos(5\pi b_3) + b_3(2x - b_3)
\end{aligned}
$$

$$
\begin{array}{lll}
z_1 = -0.2 & a_i = \frac{-1}{5\pi}\arccos\left(\frac{-0.08}{\pi^2}\right) + \frac{2}{5}(i-1) & i = 1, 2, 3\\
z_2 = 1 & b_i = \frac{1}{5\pi}\arccos\left(\frac{-0.08}{\pi^2}\right) + \frac{2}{5}(i-1) & i = 1, 2, 3
\end{array}
$$

Does the dc decomposition chosen affect?

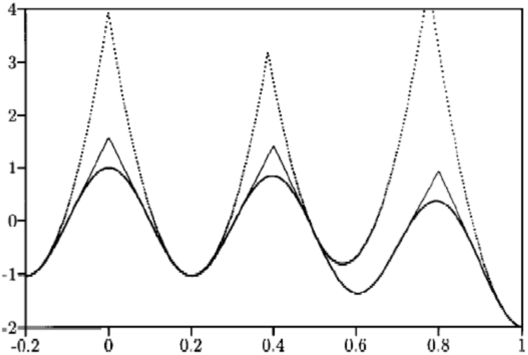


Figure 1. Comparison of envelopes I: Example function (double-wide line), Breiman–Cutler

Does this stuff work?

Table 3. Computational results for Cauchy likelihood

Sample	Sample Size	Interval	Starting Point	Number of iterations		
				Wingo	Breiman-Cutler	Algorithm 1
A	4	[3,17]	9.5	71	16	12
B	10	[2,26]	13.0	107	21	12
C	25	[4.1,2745.6]	242.5	1048	391	23
D	50	[952.1,1047.9]	999.5	-	20	11
E	100	[74.3,4905.2]	2332.1	-	1919	63

Branch and bound on a simplex S

Initialization

Take $x_1 \in S$, Set $\overline{f} = f(x_1)$, $x_* = x_1$
Construct the list $\mathcal{L}$ of simplices to be inspected with $\mathcal{L} = \{S\}$
Calculate $\underline{f}(S)$, lower bound of f on S.

Iteration $k = 1,2, \ldots$

Erase from $\mathcal{L}$ all simplices S^* with $\underline{f}(S^*) \geq \overline{f} - \varepsilon$.
Extract from $\mathcal{L}$ the simplex S^* with smallest $\underline{f}(S^*)$.
Inspect f at some $x \in S^*$ and update, if needed, $\overline{f}$.
Split S^*, and add to $\mathcal{L}$ all resulting simplices and their lower bounds.
If $\overline{f} - \underline{f} \leq \varepsilon$ then Stop.
GoTo iteration $k+1$.

Definition

$\varphi : K \subset \mathbb{R} \longrightarrow \mathbb{R}$: difference of convex monotonic (dcm) if $\varphi^1, \varphi^2 : K \subset \mathbb{R} \longrightarrow \mathbb{R}$, convex y monotonic exist s.t.

$$\varphi = \varphi^1 - \varphi^2$$

Dcm functions. Properties

If $\varphi \in \mathcal{C}^2(\mathbb{R})$, then φ : dcm

$$\begin{aligned}
\varphi(t) &= \int_{t_0}^{t} \alpha(s)\, ds + (\varphi(t_0) + \varphi'(t_0)(t - t_0)) - \int_{t_0}^{t} \beta(s)\, ds \\
\alpha(t) &= \int_{t_0}^{t} [\varphi''(s)]^{+}\, ds \\
\beta(t) &= -\int_{t_0}^{t} [\varphi''(s)]^{-}\, ds
\end{aligned}$$

Dcm functions. Properties

If $\varphi \in \mathcal{C}^2(\mathbb{R})$, then φ : dcm

$$\begin{aligned}
\varphi(t) &= \int_{t_0}^{t} \alpha(s)\,ds + \left(\varphi(t_0) + \varphi'(t_0)(t - t_0)\right) - \int_{t_0}^{t} \beta(s)\,ds \\
\alpha(t) &= \int_{t_0}^{t} \left[\varphi''(s)\right]^{+} ds \\
\beta(t) &= -\int_{t_0}^{t} \left[\varphi''(s)\right]^{-} ds
\end{aligned}$$

Dcm functions. Properties

If $\varphi = \varphi^1 - \varphi^2$, with $\varphi^1, \varphi^2 \in C^1(K)$, ($K \subset \mathbb{R}$: compact), then φ : dcm

$$\begin{aligned} \varphi(t) &= \varphi^1(t) - \varphi^2(t) \\ &= \left(\varphi^1(t) + tM\right) - \left(\varphi^2(t) + tM\right) \end{aligned}$$

Dcm functions. Properties

If $\varphi = \varphi^1 - \varphi^2$, with $\varphi^1, \varphi^2 \in C^1(K)$, $(K \subset \mathbb{R}$: compact), then φ : dcm

$$\begin{aligned}\varphi(t) &= \varphi^1(t) - \varphi^2(t) \\ &= \left(\varphi^1(t) + tM\right) - \left(\varphi^2(t) + tM\right)\end{aligned}$$

Dcm functions. Properties

Bad news (I)

The set of dcm functions is a proper subset of dc functions

$$\varphi(t) = \sqrt{t(1-t)} \qquad \forall t \in K := [0,1]$$

Bad news (II)

The set of dcm functions is not closed under sums

Dcm functions. Properties

Bad news (I)

The set of dcm functions is a proper subset of dc functions

$$\varphi(t) = \sqrt{t(1-t)} \qquad \forall t \in K := [0,1]$$

Bad news (II)

The set of dcm functions is not closed under sums

$$\varphi(t) = \sqrt{t} - e^{t} \qquad \forall t \in K := \mathbb{R}_+$$

Dcm functions. Properties

Bad news (I)

The set of dcm functions is a proper subset of dc functions

$$\varphi(t) = \sqrt{t(1-t)} \qquad \forall t \in K := [0,1]$$

Bad news (II)

The set of dcm functions is not closed under sums

$$\varphi(t) = \sqrt{t} - e^t \qquad \forall t \in K := \mathbb{R}_+$$

Dcm functions. Properties

Bad news (I)

The set of dcm functions is a proper subset of dc functions

$$\varphi(t) = \sqrt{t(1-t)} \qquad \forall t \in K := [0,1]$$

Bad news (II)

The set of dcm functions is not closed under sums

$$\varphi(t) = \sqrt{t} - e^t \qquad \forall t \in K := \mathbb{R}_+$$

Bounds for $\varphi_a = \varphi_a^1 - \varphi_a^2$, dcm

For $x = \sum_i \lambda_i v_i$, $\lambda_i \geq 0 \, \forall i$, $\sum_i \lambda_i = 1$,

- If $\varphi_a^1 : \uparrow$, then $\varphi_1^a(\| \cdot - a\|)$: convex, and

$$\begin{aligned} \varphi_a^1(\|\sum_i \lambda_i v_i - a\|_a) &\geq \varphi_a^1(\|x_0 - a\|_a) + p_a^\top (\sum_i \lambda_i v_i - x_0), \\ p_a &\in \partial \varphi_a^1(\| \cdot - a\|_a)|_{x=x_0} \end{aligned}$$

- If $\varphi_a^1 : \downarrow$, then $\varphi_a^1(d) \geq \varphi_a^1(d_0) + p_a\,(d - d_0)$, and

$$\begin{aligned} \varphi_a^1(\|\sum_i \lambda_i v_i - a\|_a) \geq \\ &\geq \varphi_a^1(\|x_0 - a\|_a) + p_a(\|\sum_i \lambda_i v_i - a\|_a - \|x_0 - a\|_a), \end{aligned}$$

$$p_a \in \partial \varphi_a^1(d)|_{d=\|x_0-a\|_a} \leq 0$$

Bounds for $\varphi_a = \varphi_a^1 - \varphi_a^2$, dcm

For $x = \sum_i \lambda_i v_i$, $\lambda_i \geq 0 \, \forall i$, $\sum_i \lambda_i = 1$,

- If $\varphi_a^1 : \uparrow$, then $\varphi_1^a(\| \cdot - a\|)$: convex, and

$$\begin{aligned} \varphi_a^1(\| \sum_i \lambda_i v_i - a\|_a) &\geq \varphi_a^1(\|x_0 - a\|_a) + p_a^\top (\sum_i \lambda_i v_i - x_0), \\ p_a &\in \partial \varphi_a^1(\| \cdot - a\|_a)|_{x=x_0} \end{aligned}$$

- If $\varphi_a^1 : \downarrow$, then $\varphi_a^1(d) \geq \varphi_a^1(d_0) + p_a\,(d - d_0)$, and

$$\begin{aligned} &\varphi_a^1(\| \sum_i \lambda_i v_i - a\|_a) \geq \\ &\geq \varphi_a^1(\|x_0 - a\|_a) + p_a(\| \sum_i \lambda_i v_i - a\|_a - \|x_0 - a\|_a), \\ p_a &\in \partial \varphi_a^1(d)|_{d=\|x_0 - a\|_a} \leq 0 \end{aligned}$$

Bounds for $\varphi_a = \varphi_a^1 - \varphi_a^2$, dcm

For $x = \sum_i \lambda_i v_i$, $\lambda_i \geq 0\, \forall i$, $\sum_i \lambda_i = 1$,

- If $\varphi_a^2 : \uparrow$, then $\varphi_a^2(\| \cdot - a\|)$: convex.
- If $\varphi_a^2 : \downarrow$, then

$$\begin{aligned} \|\sum_i \lambda_i v_i - a\|_a &\geq \|x_0 - a\| + p_a^\top \left(\sum_i \lambda_i v_i - x_0\right), \\ p_a &\in \partial(\| \cdot - a\|_a)|_{x_0} \\ \varphi_a^2(\|\sum_i \lambda_i v_i - a\|_a) &\leq \varphi_a^2\left(\|x_0 - a\| + p_a^\top(\sum_i \lambda_i v_i - x_0)\right) \end{aligned}$$

Bounds for $\varphi_a = \varphi_a^1 - \varphi_a^2$, dcm

For $x = \sum_i \lambda_i v_i$, $\lambda_i \geq 0 \, \forall i$, $\sum_i \lambda_i = 1$,

- If $\varphi_a^2 : \uparrow$, then $\varphi_a^2(\| \cdot - a\|)$: convex.
- If $\varphi_a^2 : \downarrow$, then

$$\begin{aligned} \|\sum_i \lambda_i v_i - a\|_a &\geq \|x_0 - a\| + p_a^\top \left(\sum_i \lambda_i v_i - x_0 \right), \\ p_a &\in \partial(\| \cdot - a\|_a)|_{x_0} \\ \varphi_a^2(\|\sum_i \lambda_i v_i - a\|_a) &\leq \varphi_a^2 \left(\|x_0 - a\| + p_a^\top (\sum_i \lambda_i v_i - x_0) \right) \end{aligned}$$

Bounds for $\varphi_a = \varphi_a^1 - \varphi_a^2$, dcm

$$\varphi_a^1(\sum_i \lambda_i v_i) \geq l_a(\sum_i \lambda_i v_i) \quad \text{concave}$$

$$\varphi_a^2(\sum_i \lambda_i v_i) \leq u_a(\sum_i \lambda_i v_i) \quad \text{convex}$$

$$\sum_{a \in A} \varphi_a \geq \sum_{a \in A} (l_a - u_a) \quad \text{concave}$$

$$\sum_{a \in A} \varphi_a(\sum_i \lambda_i v_i) \geq \min_i \sum_{a \in A} (l_a(v_i) - u(v_i))$$

Obnoxious facility location

JOGO (2007) bounding method

N	Iterations			Max triangles			Time (seconds)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	37	77	65,5	10	34	17,3	0,00	0,00	0,00
20	71	141	88,0	15	40	27,8	0,00	0,00	0,00
50	76	186	105,4	23	108	50,0	0,00	0,02	0,00
100	98	387	170,1	38	184	80,4	0,00	0,03	0,01
200	103	496	202,0	57	211	133,4	0,02	0,06	0,03
500	115	215	164,3	104	199	157,1	0,09	0,11	0,10
1000	158	542	317,9	196	614	334,8	0,31	0,55	0,41
2000	207	1898	567,2	272	781	527,3	1,13	3,08	1,54
5000	312	1984	609,1	399	1768	710,8	6,42	11,31	7,28
10000	368	1703	647,9	623	1511	986,8	24,11	31,94	25,75

dcm-based bounding method

N	Iterations			Max triangles			Time (seconds)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	37	77	65,5	10	34	17,3	0,00	0,00	0,00
20	71	141	88,0	15	40	27,8	0,00	0,02	0,00
50	76	186	105,4	23	108	50,0	0,00	0,02	0,00
100	98	387	170,1	38	184	80,4	0,00	0,03	0,01
200	103	496	202,0	57	211	133,4	0,02	0,06	0,03
500	115	215	164,3	104	199	157,1	0,09	0,13	0,10
1000	158	542	317,9	196	614	334,8	0,31	0,55	0,41
2000	207	1898	567,2	272	781	527,3	1,16	3,09	1,57
5000	312	1984	609,1	399	1768	710,8	6,53	11,44	7,41
10000	368	1703	647,9	623	1511	986,8	24,69	32,44	26,30

Weber problem with some negative weights

JOGO (2007) bounding method

N	Iterations			Max triangles			Time (seconds)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	115	1712	393,1	12	169	39,9	0,00	0,02	**0,00**
20	120	825	292,6	12	127	44,2	0,00	0,02	**0,00**
50	132	5689	769,0	16	479	88,3	0,00	0,11	**0,02**
100	124	481	259,0	19	119	41,2	0,00	0,02	**0,01**
200	181	633	327,5	25	99	49,3	0,02	0,06	**0,03**
500	269	1762	598,3	32	775	174,1	0,09	0,41	**0,16**
1000	248	760	406,2	27	221	79,8	0,27	0,47	**0,33**
2000	371	4233	888,3	62	1428	237,8	0,94	4,11	**1,36**
5000	389	19638	4040,7	74	7115	1335,6	4,75	44,25	**12,23**
10000	362	13761	4218,4	90	4308	1382,6	17,28	72,14	**33,07**

dcm-based bounding method

N	Iterations			Max triangles			Time (seconds)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	115	1712	393,1	12	169	39,9	0,00	0,02	**0,00**
20	120	825	292,6	12	127	44,2	0,00	0,02	**0,00**
50	132	5689	769,0	16	479	88,3	0,00	0,11	**0,02**
100	124	481	259,0	19	119	41,2	0,00	0,03	**0,01**
200	181	633	327,5	25	99	49,3	0,02	0,06	**0,03**
500	269	1762	598,3	32	775	174,1	0,09	0,39	**0,16**
1000	248	760	406,2	27	221	79,8	0,25	0,45	**0,32**
2000	371	4233	888,3	62	1428	237,8	0,92	4,06	**1,35**
5000	389	19638	4040,7	74	7115	1335,6	4,69	43,64	**12,08**
10000	362	13761	4218,4	90	4308	1382,6	17,02	71,33	**32,63**

Huff competitive location (I)

JOGO (2007) bounding method

N	Iterations Min	Iterations Max	Iterations Ave	Max triangles Min	Max triangles Max	Max triangles Ave	Time (seconds) Min	Time (seconds) Max	Time (seconds) Ave
10	1546	10272	3455,4	353	4384	1210,6	0,05	0,25	0,09
20	3261	23784	10546,3	942	10401	4202,5	0,16	1,17	0,52
50	3424	26321	9037,9	902	3033	1787,1	0,42	3,22	1,11
100	8631	38740	20839,7	1440	15614	5968,3	2,11	9,41	5,07
200	11374	31149	18604,7	2561	7331	4418,2	5,56	15,16	9,05
500	25686	225775	60419,4	2974	56600	12151,5	31,34	272,73	73,29
1000	16119	121410	49726,0	4060	36009	9533,3	40,03	294,58	121,40
2000	25003	91161	44350,3	3547	12011	6788,3	125,27	447,03	219,13
5000	23615	76870	43315,5	4086	18449	7244,7	312,31	955,73	550,42
10000	25322	113933	57053,7	4098	25665	9383,3	717,08	2852,38	1482,95

dcm-based bounding method

N	Iterations Min	Iterations Max	Iterations Ave	Max triangles Min	Max triangles Max	Max triangles Ave	Time (seconds) Min	Time (seconds) Max	Time (seconds) Ave
10	4348	38008	11682,8	928	16534	3903,5	0,08	0,67	0,20
20	10334	135627	42372,9	3518	50889	14189,7	0,36	4,70	1,46
50	10009	63998	28842,1	2706	10653	6057,8	0,86	5,44	2,45
100	26202	126039	69249,0	5218	46734	19825,6	4,44	21,36	11,73
200	39972	141281	69207,9	9098	24792	15131,0	13,50	47,63	23,35
500	84410	879358	258658,3	9687	229944	46271,5	71,16	740,73	217,87
1000	51124	588205	185917,7	11504	142366	32350,2	86,70	991,02	313,45
2000	92079	243985	145699,5	12715	34554	22735,8	312,33	822,63	492,63
5000	83212	273440	157018,1	14070	57043	25126,7	716,48	2315,28	1336,60
10000	88832	420086	202536,4	16210	87786	31930,2	1561,80	7138,22	3472,84

Both algorithms use the DC dcomposition proposed in Drezner's paper

Huff competitive location (II)

JOGO (2007) bounding method

N	Iterations Min	Iterations Max	Iterations Ave	Max triangles Min	Max triangles Max	Max triangles Ave	Time (seconds) Min	Time (seconds) Max	Time (seconds) Ave
10	1546	10272	3455,4	353	4384	1210,6	0,05	0,25	**0,09**
20	3261	23784	10546,3	942	10401	4202,5	0,16	1,17	**0,52**
50	3424	26321	9037,9	902	3033	1787,1	0,42	3,22	**1,11**
100	8631	38740	20839,7	1440	15614	5968,3	2,11	9,41	**5,07**
200	11374	31149	18604,7	2561	7331	4418,2	5,56	15,16	**9,05**
500	25686	225775	60419,4	2974	56600	12151,5	31,34	272,73	**73,29**
1000	16119	121410	49726,0	4060	36009	9533,3	40,03	294,58	**121,40**
2000	25003	91161	44350,3	3547	12011	6788,3	125,27	447,03	**219,13**
5000	23615	76870	43315,5	4086	18449	7244,7	312,31	955,73	**550,42**
10000	25322	113933	57053,7	4098	25665	9383,3	717,08	2852,38	**1482,95**

dcm-based bounding method

N	Iterations Min	Iterations Max	Iterations Ave	Max triangles Min	Max triangles Max	Max triangles Ave	Time (seconds) Min	Time (seconds) Max	Time (seconds) Ave
10	187	981	409,0	27	93	52,4	0,00	0,03	**0,01**
20	200	598	346,6	29	88	54,9	0,00	0,03	**0,01**
50	169	495	308,6	24	77	42,4	0,02	0,06	**0,03**
100	200	557	311,3	32	70	51,1	0,05	0,13	**0,08**
200	208	361	279,2	31	71	47,0	0,13	0,19	**0,16**
500	316	642	457,9	51	104	71,7	0,56	0,89	**0,71**
1000	264	866	490,1	45	149	73,0	1,45	2,72	**1,93**
2000	221	554	375,6	34	79	58,7	4,50	5,94	**5,18**
5000	214	579	356,7	37	81	54,2	24,66	28,55	**26,18**
10000	209	685	420,7	33	123	70,9	94,00	104,16	**98,51**

DCM algorithm uses an alternative DC decomposition

Huff competitive location (III)

JOGO (2007) bounding method

N	Iterations			Max triangles			Time (seconds)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	7590	191606	90873,3	3138	78502	37112,9	0,20	5,28	**2,49**
20	670	511323	130306,0	214	251744	60708,4	0,05	27,47	**7,03**
50	418	104870	31748,9	58	41895	12497,2	0,06	14,00	**4,23**
100	163	271540	62316,1	33	113022	25752,3	0,05	72,81	**16,58**
200	275	94917	26174,3	75	47849	11692,2	0,19	50,78	**13,97**
500	853	333568	81006,2	135	143653	34609,9	1,42	440,36	**107,48**
1000	3546	133319	39303,6	1132	75639	18829,2	10,42	355,70	**105,48**
2000	691	93329	24589,3	220	45317	10874,4	8,31	500,03	**135,11**
5000	5744	36640	13439,1	2334	16416	6069,4	105,25	516,77	**207,23**
10000	285	67258	11445,7	67	32693	5335,1	123,50	1895,34	**418,86**

dcm-based bounding method

N	Iterations			Max triangles			Time (seconds)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	187	981	409,0	27	93	52,4	0,00	0,03	**0,01**
20	200	598	346,6	29	88	54,9	0,00	0,03	**0,01**
50	169	495	308,6	24	77	42,4	0,02	0,06	**0,03**
100	200	557	311,3	32	70	51,1	0,05	0,13	**0,08**
200	208	361	279,2	31	71	47,0	0,13	0,19	**0,16**
500	316	642	457,9	51	104	71,7	0,56	0,89	**0,71**
1000	264	866	490,1	45	149	73,0	1,45	2,72	**1,93**
2000	221	554	375,6	34	79	58,7	4,50	5,94	**5,18**
5000	214	579	356,7	37	81	54,2	24,66	28,55	**26,18**
10000	209	685	420,7	33	123	70,9	94,00	104,16	**98,51**

Both algorithms use the alternative DC decomposition

Stochastic weigthed minimax

JOGO (2007) bounding method

N	Iterations			Max triangles			Time (seconds)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	91	154	122,3	9	16	13,4	0,00	0,02	**0,00**
20	89	166	131,7	10	26	20,0	0,00	0,02	**0,00**
50	116	182	156,7	22	39	28,4	0,02	0,03	**0,02**
100	138	235	182,1	23	35	29,1	0,03	0,05	**0,04**
200	164	237	197,4	33	41	37,6	0,09	0,13	**0,10**
500	181	297	247,4	34	49	41,0	0,36	0,47	**0,42**
1000	177	327	277,2	35	55	46,6	1,09	1,36	**1,27**
2000	199	343	292,0	30	57	44,9	3,80	4,31	**4,13**
5000	248	350	293,5	38	63	51,9	21,52	22,42	**21,91**
10000	255	475	343,5	42	67	55,3	81,64	85,61	**83,24**

dcm-based bounding method

N	Iterations			Max triangles			Time (seconds)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	96	159	127,3	11	26	19,4	0,00	0,02	**0,00**
20	97	201	139,7	15	38	27,0	0,00	0,02	**0,00**
50	128	224	172,6	25	44	33,9	0,00	0,02	**0,01**
100	147	228	191,8	27	42	36,1	0,02	0,03	**0,03**
200	187	248	214,1	37	53	43,9	0,05	0,06	**0,06**
500	202	318	264,5	38	54	43,5	0,22	0,28	**0,25**
1000	214	355	287,1	42	57	48,8	0,64	0,80	**0,72**
2000	213	340	304,6	28	65	47,6	2,08	2,36	**2,29**
5000	272	384	314,9	41	63	52,9	11,58	12,19	**11,81**
10000	261	480	352,8	43	71	57,0	43,16	45,58	**44,17**

Stochastic weigthed minimax

JOGO (2007) bounding method

N	Iterations			Max triangles			Time (seconds)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	91	154	122,3	9	16	13,4	0,00	0,02	**0,00**
20	89	166	131,7	10	26	20,0	0,00	0,02	**0,00**
50	116	182	156,7	22	39	28,4	0,00	0,03	**0,02**
100	138	235	182,1	23	35	29,1	0,03	0,05	**0,04**
200	164	237	197,4	33	41	37,6	0,08	0,11	**0,10**
500	181	297	247,4	34	49	41,0	0,36	0,47	**0,41**
1000	177	327	277,2	35	55	46,6	1,08	1,36	**1,27**
2000	199	343	292,0	30	57	44,9	3,81	4,33	**4,15**
5000	248	350	293,5	38	63	51,9	21,64	22,48	**21,99**
10000	255	475	343,5	42	67	55,3	82,05	86,00	**83,56**

dcm-based bounding method

N	Iterations			Max triangles			Time (seconds)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	92	164	125,3	11	25	19,2	0,00	0,02	**0,00**
20	97	210	143,1	15	37	26,3	0,00	0,02	**0,00**
50	129	215	168,3	25	44	33,7	0,00	0,02	**0,01**
100	151	236	191,8	26	42	35,3	0,02	0,03	**0,03**
200	180	250	210,3	37	51	42,8	0,05	0,08	**0,06**
500	197	315	259,8	37	53	42,8	0,20	0,28	**0,24**
1000	199	346	289,2	42	57	48,6	0,61	0,77	**0,70**
2000	221	346	308,0	28	61	46,3	2,02	2,28	**2,21**
5000	259	373	305,5	42	63	53,1	10,97	11,59	**11,21**
10000	263	452	352,5	42	71	56,5	41,00	43,09	**42,00**

DCM algorithm uses an alternative DC decomposition

Unserviced demand $\Phi(d)=\exp(-d)$

JOGO (2007) bounding method

N	Iterations			Max triangles			Time (seconds)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	136	402	220,9	20	53	30,0	0,00	0,02	0,00
20	132	601	209,5	18	62	27,8	0,00	0,03	0,01
50	133	410	181,5	20	55	28,3	0,02	0,03	0,02
100	116	230	165,6	20	35	25,4	0,03	0,05	0,03
200	117	235	161,6	19	33	24,6	0,06	0,09	0,08
500	116	186	135,8	17	30	22,6	0,27	0,31	0,28
1000	94	173	125,2	19	29	21,6	0,81	0,94	0,86
2000	102	125	111,3	18	22	19,9	2,98	3,06	3,02
5000	103	150	117,7	18	33	23,1	17,47	17,83	17,58
10000	83	183	115,8	18	35	23,3	67,81	69,41	68,36

dcm-based bounding method

N	Iterations			Max triangles			Time (seconds)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	132	401	215,2	18	53	29,3	0,00	0,02	0,00
20	125	601	202,5	18	61	26,7	0,00	0,02	0,00
50	125	411	174,0	18	56	26,8	0,00	0,02	0,01
100	109	222	159,3	18	34	24,0	0,00	0,03	0,01
200	113	228	154,8	17	32	23,4	0,03	0,05	0,04
500	111	184	129,9	17	28	21,4	0,09	0,13	0,10
1000	88	164	118,4	17	28	20,9	0,30	0,34	0,31
2000	92	121	106,1	16	21	18,7	1,08	1,11	1,10
5000	99	147	112,5	18	30	21,5	6,30	6,47	6,35
10000	80	175	110,5	16	33	21,7	24,41	25,06	24,62

Unserviced demand $\Phi(d)=1/(1+d)$

JOGO (2007) bounding method

N	Iterations			Max triangles			Time (seconds)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	158	940	346,3	19	109	45,0	0,00	0,02	0,00
20	170	663	265,1	24	85	35,7	0,00	0,02	0,01
50	142	705	271,5	19	107	39,9	0,00	0,06	0,02
100	132	277	185,2	21	41	27,8	0,02	0,05	0,03
200	137	303	191,5	24	46	30,7	0,06	0,13	0,09
500	133	219	154,4	20	33	25,9	0,27	0,33	0,28
1000	111	192	139,8	21	34	24,9	0,83	0,94	0,86
2000	107	137	126,7	20	27	23,5	2,94	3,03	3,00
5000	107	195	127,9	20	34	24,7	17,14	17,80	17,30
10000	104	199	129,1	21	35	24,9	66,89	68,28	67,25

dcm-based bounding method

N	Iterations			Max triangles			Time (seconds)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	151	940	335,5	18	109	42,4	0,00	0,02	0,00
20	160	662	256,5	23	86	34,4	0,00	0,02	0,00
50	129	701	262,4	19	106	38,7	0,00	0,02	0,01
100	124	267	175,6	21	41	26,5	0,00	0,02	0,01
200	129	297	180,8	21	43	28,3	0,02	0,03	0,03
500	122	214	145,2	20	33	24,1	0,08	0,09	0,08
1000	103	187	132,9	19	33	23,5	0,23	0,27	0,24
2000	101	129	116,8	18	24	20,6	0,81	0,84	0,83
5000	101	189	120,1	20	35	23,0	4,69	4,91	4,73
10000	94	192	122,1	19	34	23,9	18,20	18,69	18,34

Inventory-location problem

JOGO (2007) bounding method

N	Iterations			Max triangles			Time (seconds)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	104	333	182,3	16	38	22,4	0,00	0,02	0,00
20	102	238	147,5	14	31	19,6	0,00	0,02	0,00
50	105	212	135,3	14	29	18,5	0,00	0,02	0,01
100	97	231	131,4	14	32	19,5	0,02	0,05	0,03
200	94	129	111,0	14	21	16,6	0,05	0,08	0,06
500	94	194	136,5	15	28	20,5	0,27	0,34	0,30
1000	86	162	110,6	14	25	17,6	0,86	0,98	0,90
2000	85	133	100,5	14	23	17,5	3,14	3,31	3,20
5000	77	209	107,1	15	38	19,8	18,53	19,61	18,78
10000	70	216	112,0	14	44	22,8	72,67	75,03	73,35

dcm-based bounding method

N	Iterations			Max triangles			Time (seconds)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	98	332	180,9	15	38	22,2	0,00	0,02	0,00
20	101	235	145,7	13	30	19,0	0,00	0,02	0,00
50	103	212	133,7	14	29	18,0	0,00	0,02	0,01
100	95	231	128,8	14	32	18,8	0,00	0,02	0,01
200	87	129	109,1	14	21	16,3	0,02	0,03	0,03
500	87	194	134,2	15	27	20,1	0,09	0,14	0,12
1000	85	161	109,4	13	25	17,5	0,33	0,38	0,35
2000	84	130	99,2	14	21	17,3	1,19	1,25	1,21
5000	74	209	106,3	15	38	19,7	6,91	7,41	7,03
10000	69	216	110,7	13	44	22,6	27,05	28,11	27,35

Gradual covering

JOGO (2007) bounding method

N	Iterations			Max triangles			Time (seconds)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	120	7159	3926,2	22	1483	860,1	0,00	0,11	**0,06**
20	130	32216	4923,2	27	4962	872,0	0,00	1,02	**0,15**
50	90	1357	418,9	24	304	80,4	0,00	0,11	**0,04**
100	138	763	469,8	46	191	109,0	0,03	0,11	**0,08**
200	85	614	209,3	25	135	51,5	0,06	0,22	**0,09**
500	100	624	234,3	29	96	56,4	0,25	0,66	**0,36**
1000	164	256	198,3	41	86	57,0	0,95	1,09	**1,01**
2000	122	387	195,8	34	85	53,2	3,19	4,02	**3,42**
5000	146	310	217,5	43	105	70,9	18,67	19,98	**19,24**
10000	162	391	249,9	48	149	87,7	72,67	76,23	**74,04**

dcm-based bounding method

N	Iterations			Max triangles			Time (seconds)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	165	1298546	134099,9	35	599174	60936,2	0,00	10,27	**1,06**
20	175	33342	5118,1	43	5498	949,6	0,00	0,48	**0,08**
50	157	1447	519,4	56	290	105,7	0,00	0,06	**0,02**
100	239	836	572,8	96	198	124,8	0,02	0,06	**0,05**
200	174	681	317,4	53	157	101,2	0,05	0,11	**0,06**
500	187	843	362,0	69	173	101,2	0,14	0,39	**0,21**
1000	264	452	344,2	89	152	116,1	0,48	0,61	**0,54**
2000	218	673	353,0	79	199	131,1	1,50	2,14	**1,68**
5000	246	601	395,0	77	223	141,8	8,23	9,53	**8,78**
10000	292	742	472,8	91	249	178,6	31,50	34,75	**32,81**

The acceleration-deceleration distance

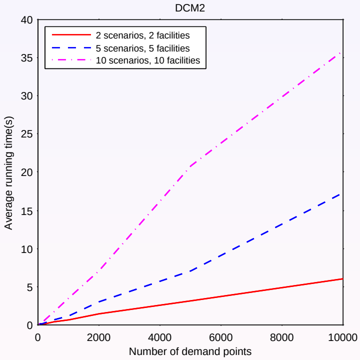
JOGO (2007) bounding method

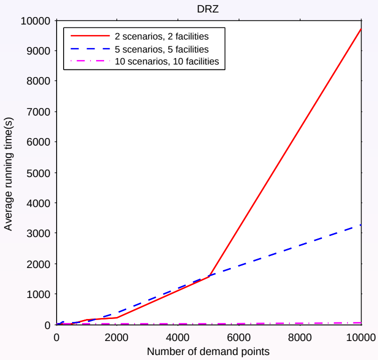
N	Iterations			Max triangles			Time (seconds)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	146	334	214,5	18	49	30,5	0,00	0,02	0,00
20	133	311	232,6	23	43	32,1	0,00	0,02	0,01
50	171	308	232,5	25	67	34,6	0,02	0,03	0,02
100	151	310	209,4	22	42	31,5	0,03	0,06	0,04
200	169	314	237,8	28	50	35,0	0,08	0,13	0,11
500	172	245	205,1	25	43	34,3	0,31	0,38	0,34
1000	164	257	193,3	24	36	29,8	0,95	1,11	1,00
2000	156	257	193,9	22	38	29,9	3,27	3,58	3,38
5000	146	247	183,8	25	39	31,3	18,38	19,16	18,67
10000	106	234	163,7	24	37	31,3	70,41	72,45	71,33

dcm-based bounding method

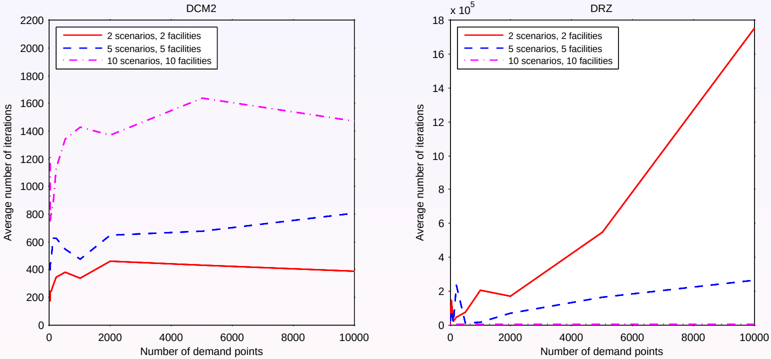
N	Iterations			Max triangles			Time (seconds)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	113	302	179,1	16	50	30,2	0,00	0,02	0,00
20	144	368	197,7	22	48	31,9	0,00	0,02	0,00
50	147	660	225,9	24	78	33,4	0,00	0,03	0,01
100	127	211	177,1	23	41	30,6	0,00	0,03	0,02
200	139	264	198,7	25	60	34,0	0,03	0,05	0,04
500	163	214	182,5	24	48	32,9	0,13	0,14	0,13
1000	143	236	170,6	21	36	27,8	0,34	0,42	0,37
2000	142	243	169,3	19	33	25,8	1,20	1,36	1,25
5000	127	236	171,5	23	41	29,9	6,77	7,14	6,92
10000	127	232	159,3	21	32	27,3	26,14	26,89	26,37

Huff with scenarios. Running times





Huff with scenarios. Number of iterations



Robust Huff location model

$$\rho(x) = \max \left\{ 0, \frac{\sum_{a \in A} \frac{\widehat{\omega}_a}{1+\beta_a \|x-a\|^\lambda} - \tau}{\|(\|x-a\|)_{a \in A}\|_*} \right\}$$

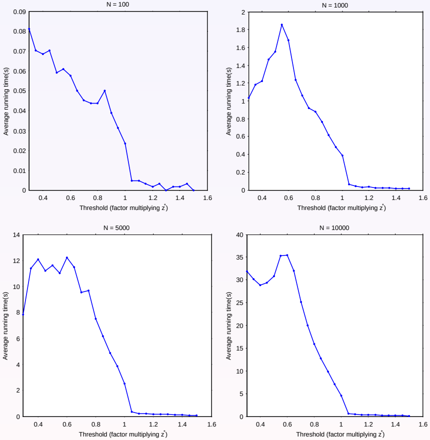
$$\rho(x) \leq \frac{U(x)}{L(x)} \leq \max_{x \in ext(T)} \frac{U(x)}{L(x)}$$

Robust Huff location model

$$\rho(x) = \max \left\{ 0, \frac{\sum_{a \in A} \frac{\widehat{\omega_a}}{1+\beta_a \|x-a\|^\lambda} - \tau}{\|(\|x-a\|)_{a\in A}\|_*} \right\}$$

$$\rho(x) \leq \frac{U(x)}{L(x)} \leq \max_{x \in ext(T)} \frac{U(x)}{L(x)}$$

Robust Huff location model



- *Thnx 4 ur attention*
- *(easy) questions? ...*
- `ecarrizosa@us.es`

- *Thnx 4 ur attention*
- *(easy) questions? ...*
- ecarrizosa@us.es

- *Thnx 4 ur attention*
- *(easy) questions? . . .*
- ecarrizosa@us.es

- *Thnx 4 ur attention*
- *(easy) questions? . . .*
- ecarrizosa@us.es

- *Thnx 4 ur attention*
- *(easy) questions? ...*
- ecarrizosa@us.es

- *Thnx 4 ur attention*
- *(easy) questions? ...*
- ecarrizosa@us.es

- *Thnx 4 ur attention*
- *(easy) questions? . . .*
- `ecarrizosa@us.es`

- *Thnx 4 ur attention*
- *(easy) questions? ...*
- ecarrizosa@us.es

- *Thnx 4 ur attention*
- *(easy) questions? ...*
- ecarrizosa@us.es

- *Thnx 4 ur attention*
- *(easy) questions? . . .*
- ecarrizosa@us.es

- *Thnx 4 ur attention*
- *(easy) questions? . . .*
- `ecarrizosa@us.es`

- *Thnx 4 ur attention*
- *(easy) questions? ...*
- ecarrizosa@us.es

- *Thnx 4 ur attention*
- *(easy) questions? ...*
- ecarrizosa@us.es

- *Thnx 4 ur attention*
- *(easy) questions? . . .*
- `ecarrizosa@us.es`

- *Thnx 4 ur attention*
- *(easy) questions? . . .*
- ecarrizosa@us.es

- *Thnx 4 ur attention*
- *(easy) questions? ...*
- ecarrizosa@us.es

- *Thnx 4 ur attention*
- *(easy) questions? . . .*
- ecarrizosa@us.es

- *Thnx 4 ur attention*
- *(easy) questions? . . .*
- ecarrizosa@us.es

- *Thnx 4 ur attention*
- *(easy) questions? ...*
- `ecarrizosa@us.es`

- *Thnx 4 ur attention*
- *(easy) questions? . . .*
- ecarrizosa@us.es

- *Thnx 4 ur attention*
- *(easy) questions? . . .*
- ecarrizosa@us.es

- *Thnx 4 ur attention*
- *(easy) questions? . . .*
- ecarrizosa@us.es

- *Thnx 4 ur attention*
- *(easy) questions? ...*
- ecarrizosa@us.es

- *Thnx 4 ur attention*
- *(easy) questions? ...*
- `ecarrizosa@us.es`

- *Thnx 4 ur attention*
- *(easy) questions? ...*
- `ecarrizosa@us.es`

- *Thnx 4 ur attention*
- *(easy) questions? ...*
- `ecarrizosa@us.es`

- *Thnx 4 ur attention*
- *(easy) questions? ...*
- `ecarrizosa@us.es`

- *Thnx 4 ur attention*
- *(easy) questions? ...*
- `ecarrizosa@us.es`

- *Thnx 4 ur attention*
- *(easy) questions? ...*
- `ecarrizosa@us.es`

- *Thnx 4 ur attention*
- *(easy) questions? ...*
- `ecarrizosa@us.es`