Discrete Competitive Facility Location Problems

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1. Problem Formulation

The facility location problem with orders:

$$\max_{(x_i)(x_{ij})} \left\{ -\sum_{i \in I} f_i x_i + \sum_{i \in I} \sum_{j \in J} p_{ij} x_{ij} \right\};$$

$$x_i + \sum_{k | i \succ_j k} x_{kj} \leq 1, \quad i \in I, \; j \in J;$$

$$x_i \geq x_{ij}, \quad i \in I, \; j \in J;$$

$$x_i, x_{ij} \in \{0, 1\}, \quad i \in I, \; j \in J.$$

$I = \{1,\ldots, m\}$: set of sites, where facilities can be located;

$J = \{1,\ldots, n\}$: set of users (customers);

p_{ij} : revenue of facility i if it services user j.

f_i : fixed cost for the opening facility i.

A relation $i \succ_j k$ (i is better than k) means that user j prefers facility i to facility k if both facilities open.

A relation $i \succcurlyeq_j k$ means that $i \succ_j k$ or $i = k$.

The three–stage making decision process

1. The Firm–Leader opens its facilities;
2. The Firm–Follower opens its facilities, taking into account the decision of the Firm–Leader.
3. Each user chooses one opened facility according his preferences and gives revenue either the Firm–Leader or the Firm–Follower.

The competitive facility location problem

$$\max_{(x_i)(x_{ij})} \left\{ -\sum_{i \in I} f_i x_i + \sum_{j \in J} \left(\sum_{i \in I} p_{ij} x_{ij} \right) \left(1 - \sum_{i \in I} \tilde{z}_{ij} \right) \right\};$$

$$x_i + \sum_{k | i \succ_j k} x_{kj} \le 1, \quad i \in I, \quad j \in J; \qquad (L)$$

$$x_i \ge x_{ij}, \quad i \in I, \quad j \in J;$$

$$x_i, x_{ij} \in \{0, 1\}, \quad i \in I, \quad j \in J;$$

$((\tilde{z}_i), (\tilde{z}_{ij}))$ is optimal solution of the follower problem:

$$\max_{(z_i)(z_{ij})} \left\{ -\sum_{i\in I} g_i x_i + \sum_{j\in J}\sum_{i\in I} p_{ij} z_{ij} \right\};$$

$$x_i + z_i + \sum_{k|i \succ_j k} z_{kj} \le 1, \quad i \in I, \quad j \in J; \qquad (F)$$

$$z_i \ge z_{ij}, \quad i \in I, \quad j \in J;$$

$$z_i, z_{ij} \in \{0, 1\}, \quad i \in I, \quad j \in J.$$

f_i : fixed costs for the opening facility i by Firm–Leader;

g_i : fixed costs for the opening facility i by Firm–Follower.

$((\tilde{z}_i),(\tilde{z}_{ij}))$ is optimal solution of the problem:

$$\max_{(z_i),(z_{ij})} \sum_{j \in J} \sum_{i \in I} p_{ij} z_{ij}$$

$$x_i + z_i + \sum_{k | i \succ_j k} z_{kj} \leq 1, \quad i \in I, \quad j \in J;$$

$$z_i \geq z_{ij}, \quad i \in I, \quad j \in J; \qquad (F_1)$$

$$\sum_{j \in J} p_{ij} z_{ij} \geq g_i z_i, \quad i \in I;$$

$$z_i, z_{ij} \in \{0, 1\}, \quad i \in I, \quad j \in J.$$

2. Optimal Solution

Notation:

$X = ((x_i), (x_{ij}))$ is a feasible solution of problem L;

$\tilde{Z} = ((\tilde{z}_i), (\tilde{z}_{ij}))$ is an optimal solution of problem F;

$(X, \tilde{Z})$ is a feasible solution of problem (L, F);

$L(X, \tilde{Z})$ is an objective function of problem (L, F)

$$L(X,\tilde{Z}) = -\sum_{i\in I} f_i x_i + \sum_{j\in J}\left(\sum_{i\in I} p_{ij} x_{ij}\right)\left(1 - \sum_{i\in I}\tilde{z}_{ij}\right).$$

X is a feasible solution of problem L .

Definition. An optimal solution $\bar{Z}$ of problem F is called *an optimal non–cooperative solution* of problem F, if $L(X,\bar{Z}) \le L(X,\tilde{Z})$ for any optimal solution $\tilde{Z}$ of problem F.

Definition. A feasible solution $(X,\bar{Z})$ of problem (L, F) is called *a feasible non–cooperative solution* of problem (L, F), if $\bar{Z}$ is an optimal non–cooperative solution of problem F.

Definition. A feasible non–cooperative solution $(X^*,\bar{Z}^*)$ of problem (L, F) is called *an optimal non–cooperative solution* of problem (L, F), if $L(X^*,\bar{Z}^*) \ge L(X,\bar{Z})$ for any feasible non–cooperative solution $(X,\bar{Z})$ of problem (L, F).

X is a feasible solution of problem L.

Step 1. Solve problem F and compute an optimal value F^* of the objective function.

Step 2. Solve the following auxiliary problem:

$$\min_{(z_i)(z_{ij})} \sum_{j \in J} \left(\sum_{i \in I} p_{ij} x_{ij} \right) \left(1 - \sum_{i \in I} z_{ij} \right);$$

$$x_i + z_i + \sum_{k | i \succ_j k} z_{kj} \le 1, \quad i \in I, \quad j \in J;$$

$$z_i \ge z_{ij}, \quad i \in I, \quad j \in J;$$

$$-\sum_{i \in I} g_i z_i + \sum_{j \in J} \sum_{i \in I} p_{ij} z_{ij} \ge F^*;$$

$$z_i,\, z_{ij} \in \{0, 1\}, \quad i \in I, \quad j \in J.$$

$\bar{Z}$ is an optimal solution of the problem.

$(X, \bar{Z})$ is a feasible non–cooperative solution of problem (L, F).

3. Upper Bound for the Optimal Value of the Objective Function

Conjecture. The requirement on p_{ij}, $i \in I$, is that for every $j \in J$ if $i \succ_j k$ then $p_{ij} \geq p_{kj}$.

Example.

$$p_{ij} = \begin{cases} b_j & \text{if } i \succcurlyeq_j i(j), \\ 0, & \text{otherwise,} \end{cases}$$

where $i(j) \in I$ is a given value for every $j \in J$.

For every $j \in J$ determine a set $I_j \subset I$ as follows.

For given $i \in I$ consider sets

$$N(i) = \{k \in I \mid k \succ_j i\},$$

$$J(i) = \{s \in J \mid i \succcurlyeq_s k, \quad k \notin N(i)\}.$$

Construct a set $J(k, i) = \{s \in J \mid k \succcurlyeq_s i\}$ for every $k \in N(i)$.

Definition. $i \in I_j$, if $g_k > \sum_{s \in J(k,i)} p_{ks}$ for every $k \in N(i)$.

Lemma 1. Let $(X, \bar{Z})$ be a feasible non–cooperative solution. If $p_{i_0 j_0} x_{i_0 j_0} > 0$ for some $j_0 \in J$, $i_0 \notin I_{j_0}$ then $\sum_{i \in I} \bar{z}_{ij_0} = 1$.

Lemma 2. Let $(X, \bar{Z})$ be a feasible non–cooperative solution. For any $j_0 \in J$ the following inequality holds

$$\left(\sum_{i \in I} p_{ij_0} x_{ij_0}\right)\left(1 - \sum_{i \in I} \bar{z}_{ij_0}\right) \le \max_{i \in I_{j_0}} p_{ij_0} x_i .$$

Define matrix (h_{ij}), $i \in I$, $j \in J$, setting

$$h_{ij} = \begin{cases} 1 \text{ if } i \in I_j, \\ 0, \quad \text{othrwise.} \end{cases}$$

Theorem. Value

$$\max_{(x_i)} \left\{ -\sum_{i \in I} f_i x_i + \sum_{j \in J} \max_{i \mid x_i = 1} p_{ij} h_{ij} \right\}$$

is an upper bound of values of the objective function of problem (L, F) on feasible non–cooperative solutions.

Algorithm for computing upper bound

Stage 1. For every $j \in J$ construct sets $I_j \subset I$.

Stage 2. Solve following facility location problem:

$$\max_{(x_i)(x_{ij})} \left\{ -\sum_{i \in I} f_i x_i + \sum_{j \in J} \sum_{i \in I} p_{ij} h_{ij} x_{ij} \right\};$$

$$\sum_{i \in I} x_{ij} = 1, \quad j \in J;$$

$$x_i \geq x_{ij}, \quad i \in I, \quad j \in J;$$

$$x_i, x_{ij} \in \{0,1\}, \quad i \in I, \quad j \in J.$$

4. Local Search Algorithm

$X = ((x_i), (x_{ij})) \leftrightarrow x = (x_i)$

$\bar{Z} = ((\bar{z}_i), (\bar{z}_{ij})) \leftrightarrow \bar{z} = (\bar{z}_i)$

$(X, \bar{Z}) \leftrightarrow (x, \bar{z}) \leftrightarrow x$

$N(x) = \{x' \mid d(x, x') \leq 2, \ |d(0, x) - d(0, x')| \leq 1\}$

$N_0(x) \subset N(x)$

$N_0(x) = \{x^k = (x_i^k), \quad k \in I\}$

$\Delta_i(x)$ denotes the “viability” of facility $i \in I$ relative to x;

$\Delta_i(x, \bar{z})$ denotes the “viability” of facility $i \in I$ relative to $(x, \bar{z})$;

$(x,\bar{z}), x=(x_i), \bar{z}=(\bar{z}_i)$ is a current solution;

$I_0(x)=\{i\in I \mid x_i=1\}$.

For $k\in I$ we obtain a vector $x^k=(x_i^k)$ by the following way.

If $i\neq k$ then $x_i^k=x_i$.

If $x_k=1$ then $x_k^k=0$.

If $x_k=0$ then $x_k^k=1$. Moreover if $\Delta_k(x^k)\geq 0$ then set $x_{i_0}^k=0$

Where $i\in I_0(x)$ such that $\min\limits_{i\in I_0(x)}\Delta_i(x^k)=\Delta_{i_0}(x^k)<0$

$(\min\limits_{i\in I_0(x)}\Delta_i(x^k,\bar{z})=\Delta_{i_0}(x^k,\bar{z})<0)$.

Algorithm

Initial step. Given initial solution x^*. Determine an optimal non–cooperative solution $\bar{z}$ and value L of the objective function of problem L.

Main step. Given a current solution $(x,\bar{z})$ and value L of the objective function. For every $k \in I$ design a vector x^k and find an optimal non–cooperative solution $\bar{z}$ and value L^k of the objective function.

If $\max_{k \in I} L^k = L^{k_0} > L$ then set $x = x^{k_0}$, $\bar{z} = \bar{z}^{k_0}$, $L = L^{k_0}$ and repeat the main step. Otherwise STOP, $(x,\bar{z})$ is the required solution.